## Homework 22 : Due Wednesday, November 11

**Problem 1:** Chapter 12, #6abc

**Problem 2:** Chapter 12, #24

**Problem 3:** This problem provides another proof of Cauchy's Theorem. Let $G$ be a group and suppose that $p$ is a prime which divides $|G|$. Let

$$X = \{(a_1, a_2, \ldots, a_{p-1}, a_p) \in G^p : a_1 a_2 \cdots a_{p-1} a_p = e\}$$

i.e. $X$ consists of all $p$-tuples of elements of $G$ such that when you multiply them in the given order you get the identity.

a. Give four examples of elements of $X$ in the special case when $G = S_3$ and $p = 3$.

b. Show that $|X| = |G|^{p-1}$.

c. Show that if $(a_1, a_2, \ldots, a_{p-1}, a_p) \in X$, then $(a_2, a_3, \ldots, a_p, a_1) \in X$. It follows that any cyclic shift of an element of $X$ remains in $X$.

Let $H$ be the subgroup of $S_p$ generated by the element $(12 \ldots p)$, so $|H| = p$. Let $H$ act on $X$ by permuting the elements, i.e. if $\sigma \in H$ and $(a_1, a_2, \ldots, a_{p-1}, a_p) \in X$, then

$$\sigma * (a_1, a_2, \ldots, a_{p-1}, a_p) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(p-1)}, a_{\sigma(p)})$$

In other words, $(12 \ldots p)$ shifts an element in $X$ to the left one (as in part b), $(12 \ldots p)^2$ shifts to the left 2, etc.

d. Show that this is indeed an action of $H$ on $X$.

e. Notice that $(e, e, \ldots, e, e) \in X_H$. Show that $X_H$ has at least one other element. *Hint:* Use Problem 2.

f. Conclude that $G$ has an element of order $p$.