Homework 20 : Due Wednesday, November 4

Problem 1: Using the Fundamental Theorem of Finite Abelian Groups, find all abelian groups up to isomorphism of each of the following orders:
a. Order 343.
b. Order 200.
c. Order 900.

Problem 2: We know from the Fundamental Theorem of Finite Abelian groups that every abelian group is isomorphic to a direct product of cyclic groups of the form $\mathbb{Z}_{p^\alpha}$. Determine those cyclic groups explicitly for each of the following.
a. $U(9)$
b. $U(15)$
Hint: You don't need to construct explicit isomorphisms. You know it must be isomorphic to one of the them, so you can use process of elimination.

Problem 3: In class, we proved that $\mathbb{Z}_p$ is an abelian simple group for all primes p. In this problem, we show that these are the only abelian simple groups (up to isomorphism). *Without using the Fundamental Theorem of Finitely Generated Abelian Groups*, show the following.
a. Suppose that G is a finite abelian group of composite (i.e. nonprime) order $n > 1$. Show that G is not simple.
b. Suppose that G is an infinite abelian group. Show that G is not simple.
Hint: Consider the various cyclic subgroups of G.

Problem 4: Let G be the set of all functions $f : \mathbb{N} \to \mathbb{Z}_2$ (you can think of elements of G as infinite sequences of 0's and 1's). Define an operation $+$ on G by letting $f + g$ be the function where $(f + g)(n) = f(n) + g(n)$ for all $n \in \mathbb{N}$.
a. Show that G with the operation $+$ is an abelian group.
b. Show that every nonidentity element of G has order 2.
c. Show that G is not finitely generated.
d. (Ungraded Challenge Problem) Show that $G \times G \cong G$.