Homework 16 : Due Friday, October 16

Problem 1: Let G be a group. Suppose that H and K are both normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G (you already know from a previous homework that it is a subgroup).

Problem 2: Let G be a group. Suppose that H is an arbitrary subgroup of G and N is a normal subgroup of G. Show that the set $HN = \{hn : h \in H, n \in N\}$ is a subgroup of G.

Problem 3: Suppose that H is a subgroup of G and that $[G : H] = 2$. Show that H is normal in G.

Problem 4: Which of the following subgroups of D_4 are normal? Explain.
a. $\{e, r, r^2, r^3\}$
b. $\{e, r^2\}$
c. $\{e, s\}$
d. $\{e, rs\}$

Problem 5: In class and in the book, we saw that if G is a group and N is a normal subgroup of G, then the operation

$$(aN) \cdot (bN) = (ab)N$$

is well-defined on left cosets. Here, we prove the converse.

Suppose then that G is a group and H is a subgroup of G. Suppose that the operation

$$(aH) \cdot (bH) = (ab)H$$

is well-defined on left cosets, i.e. whenever $aH = cH$ and $bH = dH$, we have $(ab)H = (cd)H$. Prove that H is a normal subgroup of G.

Hint: We know that it suffices to show that $gHg^{-1} \subseteq H$ for all $g \in G$. For a given $g \in G$ and $h \in H$, consider $a = e$, $b = g^{-1}$, $c = h$, and $d = g^{-1}$.