Metric Spaces Crash Course

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August 31, 2023

1 Metric Spaces

1.1 Definition and Examples

Definition 1.1. *A* metric space *is a nonempty set X together with a function $d\colon X^2 \to \mathbb{R}$ having the following properties:*

1. $d(x,x) = 0$ *for all* $x \in X$*, and* $d(x,y) > 0$ *whenever* $x, y \in X$ *with* $x \neq y$.
2. $d(x,y) = d(y,x)$ *for all* $x, y \in X$.
3. $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x, y, z \in X$.

Examples:

1. $X = \mathbb{R}$ with $d(x,y) = |x-y|$.
2. $X = \mathbb{R}^n$ with $d((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)) = \max\{|x_1-y_1|,|x_2-y_2|,\dots,|x_n-y_n|\}$.
3. $X = \mathbb{R}^n$ with $d((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)) = |x_1-y_1|+|x_2-y_2|+\cdots+|x_n-y_n|$.
4. $X = \mathbb{R}^n$ with $d((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$.
5. $X = C[0,1]$ is the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$ with $d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$.
6. $X = C[0,1]$ is the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$ with $d(f,g) = \int_0^1 |f(x)-g(x)|\ dx$.
7. $X = C[0,1]$ is the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$ with $d(f,g) = \sqrt{\int_0^1 (f(x)-g(x))^2\ dx}$.
8. Let $X = \{0,1\}^{\mathbb{N}}$ be the set of all infinite sequences of 0s and 1s. Alternatively, one can define X to be the set of all functions $f\colon \mathbb{N} \to \{0,1\}$. Define d by letting $d((x_0,x_1,x_2,\dots),(y_0,y_1,y_2,\dots)) = \frac{1}{2^m}$, where m is the least n with $x_n \neq y_n$, and letting it be 0 if no such n exists.
9. $X = [0,1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ and
$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \min\{y-x, x+1-y\} & \text{if } x < y, \\ \min\{x-y, y+1-x\} & \text{if } y < x. \end{cases}$$
10. Let X be any set, and define
$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 1.2. *Let (X, d) be a metric space, let $\langle x_n \rangle$ be a sequence of points from X, and let $\ell \in X$. We say that $\langle x_n \rangle$* converges to ℓ, *and write $\lim_{n\to\infty} x_n = \ell$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}^+$ such for all $n \geq N$, we have $d(x_n, \ell) < \varepsilon$.*

Definition 1.3. *Let (X, d) be a metric space and let $\langle x_n \rangle$ be a sequence of points from X. We say that $\langle x_n \rangle$* converges *if there exists an $\ell \in X$ such that $\langle x_n \rangle$ converges to ℓ. If $\langle x_n \rangle$ does not converge, we say that $\langle x_n \rangle$* diverges.

Proposition 1.4. *Suppose that $\langle x_n \rangle$ is a convergent sequence in a metric space (X, d). There exists a unique $\ell \in \mathbb{R}$ with $\lim_{n\to\infty} x_n = \ell$.*

Proof. The existence of an ℓ is immediate from the definition. For uniqueness, suppose that $\langle x_n \rangle$ converges to both ℓ and m. We show that $d(\ell, m) < \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon > 0$ be arbitrary. Since $\langle x_n \rangle$ converges to ℓ, we can fix $N_1 \in \mathbb{N}^+$ such that $d(x_n, \ell) < \frac{\varepsilon}{2}$ for all $n \geq N_1$. Since $\langle x_n \rangle$ converges to m, we can fix $N_2 \in \mathbb{N}^+$ such that $|x_n - m| < \frac{\varepsilon}{2}$ for all $n \geq N_2$. Consider $n = \max\{N_1, N_2\}$. We then have

$$\begin{aligned}
d(\ell, m) &\leq d(\ell, x_n) + d(x_n, m) \\
&= d(x_n, \ell) + d(x_n, m) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{(since } n \geq N_1 \text{ and } n \geq N_2\text{)} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that $d(\ell, m) < \varepsilon$ for all $\varepsilon > 0$. Thus, we must have $d(\ell, m) = 0$, so by Property 1 of a metric space, it follows that $\ell = m$. □

Definition 1.5. *Let (X, d) be a metric space and let $\langle x_n \rangle$ be a sequence of points from X, and let $\ell \in X$. We say that $\langle x_n \rangle$ is a* Cauchy sequence *if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}^+$ such for all $m, n \geq N$, we have $d(x_m, x_n) < \varepsilon$.*

Proposition 1.6. *In a metric space (X, d), every convergent sequence is a Cauchy sequence.*

Proof. Suppose that $\langle x_n \rangle$ is a convergent sequence from X. Fix $\ell \in X$ with $\lim_{n\to\infty} x_n = \ell$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = \ell$, we can fix $N \in \mathbb{N}^+$ such that $d(x_n, \ell) < \frac{\varepsilon}{2}$ for all $n \geq N$. For any $m, n \geq N$, we then have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, \ell) + d(\ell, x_n) \\
&\leq d(x_m, \ell) + d(x_n, \ell) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{(since } n, m \geq N\text{)} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\langle x_n \rangle$ is a Cauchy sequence. □

In a metric space, the converse of this statement is *not* generally true.

Definition 1.7. *A metric space (X, d) is* complete *if every Cauchy sequence converges.*

The set $\mathbb{R}$ with its usual metric is complete. However, the set $\mathbb{Q}$ with the usual metric (inherited from $\mathbb{R}$) is *not* complete. Intuitively, a complete metric space is one that does not have any “holes” or “gaps”.

1.2 Interior, Closure, and Limit Points

Definition 1.8. *Let (X,d) be a metric space. Given $x \in X$ and $r \in \mathbb{R}$, we let $B_r(x) = \{y \in X : d(x,y) < r\}$. The set $B_r(x)$ is called the* ball of radius r *around x.*

Definition 1.9. *Let (X,d) be a metric space, let $A \subseteq X$ and let $x \in X$.*

- *We say that x is an* interior point *of A if there exists $\varepsilon > 0$ with $B_\varepsilon(x) \subseteq A$.*
- *We say that x is a* closure point *of A if for all $\varepsilon > 0$, we have $B_\varepsilon(x) \cap A \neq \emptyset$.*
- *We say that x is a* limit point *of A if for all $\varepsilon > 0$, the set $B_\varepsilon(x) \cap A$ contains at least one point distinct from x.*

Intuitively, a point $x \in X$ is an interior point of A is it sits "comfortably" inside A because we can fatten up x to a whole (potentially very small) ball of points around x that completely sits inside A. A point x is a closure point of A if it is very friendly with points of A in the sense that there are points of A that are arbitrarily close to x. Finally, the difference between a limit point and a closure point is that we do not allow the situation where x is a hermit, i.e. we require that every neighborhood of x contains points of A, but we do not allow x to serve as such a "close" point.

Notation 1.10. *Let (X,d) be a metric space, and let $A \subseteq X$.*

- *We let $\mathsf{int}(A) = \{x \in X : x \text{ is an interior point of } A\}$, and call $\mathsf{int}(A)$ the* interior *of A. Some sources use the notation A° for the interior of A.*
- *We let $\mathsf{cl}(A) = \{x \in X : x \text{ is a closure point of } A\}$, and call $\mathsf{cl}(A)$ the* closure *of A. Some sources use the notation $\overline{A}$ for the closure of A.*

To get a feel for these definitions, let's consider some examples. Working in $\mathbb{R}$ with the usual metric, consider the set

$$A = \{x \in \mathbb{R} : 0 \leq x < 1\} \cup \{2\} \cup \left\{4 - \frac{1}{n} : n \in \mathbb{N}^+\right\}.$$

Although we will not work through careful proofs now, we have the following:

- $\mathsf{int}(A) = (0,1)$: Intuitively, we can expand any element of $(0,1)$ to a neighborhood that will still be a subset of A, but this will not work for 0 or any of the other points.
- $\mathsf{cl}(A) = [0,1] \cup \{2,4\} \cup \left\{4 - \frac{1}{n} : n \in \mathbb{N}^+\right\}$: All of the points of A are closure points, and now we include both 1 and 4 because there are points of A that are arbitrarily close to these.
- $\{x \in \mathbb{R} : x \text{ is a limit point of } A\} = [0,1] \cup \{4\}$: Here we can find points of A arbitrarily close to 4 that are distinct from 4, but the same can not be said of the points 2 and $4 - \frac{1}{n}$.

If we instead consider the subset $\mathbb{Q} \subseteq \mathbb{R}$, we obtain the following:

- $\mathsf{int}(\mathbb{Q}) = \emptyset$: We know that every open interval contains irrationals.
- $\mathsf{cl}(\mathbb{Q}) = \mathbb{R}$: Given any $x \in \mathbb{R}$ and any $\varepsilon > 0$, we know that the open interval $B_\varepsilon(x)$ contains a rational by the Density of $\mathbb{Q}$ in $\mathbb{R}$.
- $\{x \in \mathbb{R} : x \text{ is a limit point of } \mathbb{Q}\} = \mathbb{R}$: Using the Density of $\mathbb{Q}$ and $\mathbb{R}$ a couple of times, given any $x \in \mathbb{R}$ and any $\varepsilon > 0$, we know that the open interval $B_\varepsilon(x)$ contains several rationals, so will contain a rational distinct from x.

We start with some simple properties of the interior and closure. The proofs are exercises.

Proposition 1.11. *Let (X, d) be a metric space. For every $A \subseteq X$, we have $\mathsf{int}(A) \subseteq A \subseteq \mathsf{cl}(A)$.*

Proposition 1.12. *Let (X, d) be a metric space. If $A \subseteq B \subseteq X$, then $\mathsf{int}(A) \subseteq \mathsf{int}(B)$ and $\mathsf{cl}(A) \subseteq \mathsf{cl}(B)$.*

A point x is in $\mathsf{cl}(A)$ if it can be well-approximated by points of A (because we can find points of A that are arbitrarily close to x). Another way to say that x can be well-approximated by points of A is to say that we can find a sequence from A that converges to x. We now prove that these two concepts coincide.

Proposition 1.13. *Let (X, d) be a metric space, let $A \subseteq X$ and let $x \in X$. The following are equivalent:*

1. *$x \in \mathsf{cl}(A)$.*
2. *There exists a sequence $\langle a_n \rangle$ with $a_n \in A$ for all $n \in \mathbb{N}^+$ such that $\langle a_n \rangle$ converges to x.*

Proof. First, we assume (1), i.e. that $x \in \mathsf{cl}(A)$. We define a sequence $\langle a_n \rangle$ as follows. Given $n \in \mathbb{N}^+$, we have that $\frac{1}{n} > 0$, so we know that $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$, and we choose a_n to be some element of the nonempty set $B_{\frac{1}{n}}(x) \cap A$. Notice that $a_n \in A$ for all $n \in \mathbb{N}^+$ by definition. We show that $\langle a_n \rangle$ converges to x. Let $\varepsilon > 0$. Fix $N \in \mathbb{N}^+$ with $N > \frac{1}{\varepsilon}$. Now given any $n \geq N$, we know that $\frac{1}{n} \leq \frac{1}{N}$, so

$$\begin{aligned}
d(a_n, x) &< \frac{1}{n} && \text{(since } a_n \in B_{\frac{1}{n}}(x)\text{)} \\
&\leq \frac{1}{N} \\
&< \varepsilon.
\end{aligned}$$

Therefore, $\langle a_n \rangle$ converges to x.

Conversely, assume (2), and fix a sequence $\langle a_n \rangle$ with $a_n \in A$ for all $n \in \mathbb{N}^+$ such that $\langle a_n \rangle$ converges to x. We show that $x \in \mathsf{cl}(A)$. Let $\varepsilon > 0$. Since $\langle a_n \rangle$ converges to x, we can fix $N \in \mathbb{N}^+$ such that for all $n \geq N$, we have $d(a_n, x) < \varepsilon$. In particular, we have $d(a_N, x) < \varepsilon$, and hence $a_N \in B_\varepsilon(x)$. Since $a_N \in A$ by assumption, we conclude that $B_\varepsilon(x) \cap A$ is nonempty. We have shown that $B_\varepsilon(x) \cap A \neq \emptyset$ for every $\varepsilon > 0$, so it follows that $x \in \mathsf{cl}(A)$. □

For limit points, there is a similar characterization where we change the sequence to require that we never use the point x itself.

Proposition 1.14. *Let (X, d) be a metric space, let $A \subseteq X$ and let $x \in X$. The following are equivalent:*

1. *x is a limit point of A.*
2. *There exists a sequence $\langle a_n \rangle$ with $a_n \in A$ and $a_n \neq x$ for all $n \in \mathbb{N}^+$ such that $\langle a_n \rangle$ converges to x.*

Proof. Adapt the proof of Proposition 1.13. The details are left as an exercise. □

1.3 Open and Closed Sets

Let (X, d) be a metric space. Recall from Proposition 1.11 that

$$\mathrm{int}(A) \subseteq A \subseteq \mathrm{cl}(A)$$

for every $A \subseteq X$. Sets A that achieve equality on one of these containments are given a special name.

Definition 1.15. *Let (X, d) be a metric space, and let $A \subseteq X$.*

1. *We say that A is* open *if $A \subseteq \mathsf{int}(A)$, i.e. if every element of A is an interior point of A. Notice that this is equivalent to saying that $A = \mathsf{int}(A)$.*

2. *We say that A is* closed *if $\mathsf{cl}(A) \subseteq A$, i.e. if every closure point of A is an element of A. Notice that this is equivalent to saying that $A = \mathsf{cl}(A)$.*

Working in $\mathbb{R}$ with the usual metric, every open interval (c, d) is an open set and every closed interval $[c, d]$ is a closed set (these are good exercises). Notice that $\emptyset$ and $\mathbb{R}$ each trivially satisfy the two definitions, so $\emptyset$ and $\mathbb{R}$ are both open and closed. There also exist sets that are neither open nor closed, such as $[c, d)$ and $\mathbb{Q}$. There exist many other sets aside from intervals that are either open or closed. In order to construct some examples, we first show that the collection of open (resp. closed) sets are closed under finite unions and finite intersections.

Proposition 1.16. *Let (X, d) be a metric space, and let $A_1, A_2, \ldots, A_n \subseteq X$.*

1. *If each A_i is open, then both $A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1 \cap A_2 \cap \cdots \cap A_n$ are open.*
2. *If each A_i is closed, then both $A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1 \cap A_2 \cap \cdots \cap A_n$ are closed.*

Proof.

1. Assume that each A_i is open.
   - $A_1 \cup A_2 \cup \cdots \cup A_n$ is open: Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$ be arbitrary. Since $x \in A_1 \cup A_2 \cup \cdots \cup A_n$, we can fix an i with $x \in A_i$. Since A_i is open, we know that $x \in \mathsf{int}(A_i)$, so we can fix $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_i$. Since $A_i \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$, it follows that $B_\varepsilon(x) \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$. Therefore, $x \in \mathsf{int}(A_1 \cup A_2 \cup \cdots \cup A_n)$.
   - $A_1 \cap A_2 \cap \cdots \cap A_n$ is open: Let $x \in A_1 \cap A_2 \cap \cdots \cap A_n$ be arbitrary. For each i, we have that $x \in A_i$ and A_i is open, so we can fix $\varepsilon_i > 0$ with $B_{\varepsilon_i}(x) \subseteq A_i$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, and notice that $\varepsilon > 0$. As $\varepsilon_i \leq \varepsilon$ for all i, it follows that $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq A_i$ for all i, so $B_\varepsilon(x) \subseteq A_1 \cap A_2 \cap \cdots \cap A_n$. Therefore, $x \in \mathsf{int}(A_1 \cap A_2 \cap \cdots \cap A_n)$.
2. Assume that each A_i closed.
   - $A_1 \cup A_2 \cup \cdots \cup A_n$ is closed: Let $x \in \mathsf{cl}(A_1 \cup A_2 \cup \cdots \cup A_n)$ be arbitrary. By Proposition 1.13, we can fix a sequence $\langle a_m \rangle$ with $a_m \in A_1 \cup A_2 \cup \cdots \cup A_n$ for all $m \in \mathbb{N}^+$ and such that $\langle a_m \rangle$ converges to x. Since $\mathbb{N}^+$ is infinite, there must exist an i such that $\{m \in \mathbb{N}^+ : a_m \in A_i\}$ is infinite. Fix such an i. We can take the values of m from the set $\{m \in \mathbb{N}^+ : a_m \in A_i\}$ to extract a subsequence of $\langle a_m \rangle$ consisting of elements of A_i. Since a subsequence of a convergent sequence must converge to the same limit (this is a nice exercise), it follows from Proposition 1.13 that $x \in \mathsf{cl}(A_i)$. Since A_i is closed, we conclude that $x \in A_i$, and hence $x \in A_1 \cup A_2 \cup \cdots \cup A_n$.
   - $A_1 \cap A_2 \cap \cdots \cap A_n$ is closed: Let $x \in \mathsf{cl}(A_1 \cap A_2 \cap \cdots \cap A_n)$ be arbitrary. We first show that $x \in \mathsf{cl}(A_i)$ for all i. Let $\varepsilon > 0$ be arbitrary. Since $x \in \mathsf{cl}(A_1 \cup A_2 \cup \cdots \cup A_n)$, we know that $B_\varepsilon(x) \cap (A_1 \cap A_2 \cap \cdots \cap A_n) \neq \emptyset$. Now $B_\varepsilon(x) \cap (A_1 \cap A_2 \cap \cdots \cap A_n)$ is a subset $B_\varepsilon(x) \cap A_i$ for all i, so $B_\varepsilon(x) \cap A_i \neq \emptyset$ for all i. Since $\varepsilon > 0$ was arbitrary, we conclude that $x \in \mathsf{cl}(A_i)$ for all i. Recall that each A_i is closed, so it follows that $x \in A_i$ for all i, and hence $x \in A_1 \cap A_2 \cup \cdots \cap A_n$.

□

We can use Proposition 1.16 to provide other examples of open and closed sets in $\mathbb{R}$ under the usual metric. For example, since $(0, 1)$ and $(2, 3)$ are both open, it follows that $(0, 1) \cup (2, 3)$ is open. Since $[1, 2]$ and $[2, 3]$ are both closed, we conclude that $\{2\} = [1, 2] \cap [2, 3]$ is closed. More generally, a straightforward argument shows that in any metric space (X, d), the one-element set $\{x\}$ is closed for each $x \in X$. Using Proposition 1.16, any finite subset of a metric space is closed.

Proposition 1.17. *Let (X, d) be a metric space. For any $x \in X$ and $\varepsilon > 0$, the set $B_\varepsilon(x)$ is open.*

Proof. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Let $y \in B_\varepsilon(x)$ be arbitrary. By definition, we then have that $d(x, y) < \varepsilon$. Let $\delta = \varepsilon - d(x, y) > 0$. We claim that $B_\delta(y) \subseteq B_\varepsilon(x)$. To see this, let $z \in B_\delta(y)$ be arbitrary. We then have that $d(y, z) < \delta$, so

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &< d(x,y) + \delta \\ &= d(x,y) + (\varepsilon - d(x,y)) \\ &= \varepsilon, \end{aligned}$$

and hence $z \in B_\varepsilon(x)$. Since $z \in B_\delta(y)$ was arbitrary, it follows that $B_\delta(y) \subseteq B_\varepsilon(x)$, so y is an interior point of $B_\varepsilon(x)$. Therefore, $B_\varepsilon(x)$ is open. □

Our next result provides another way to construct open and closed sets. At first, it might appear immediate from the definition that $\mathsf{int}(A)$ is always an open set. However, there is some real subtlety here. Recall that to show that a set B is open, we have to prove that $B \subseteq \mathsf{int}(B)$. Thus, to show that $\mathsf{int}(A)$ is open, we have to prove that $\mathsf{int}(A) \subseteq \mathsf{int}(\mathsf{int}(A))$. Similarly, to show that $\mathsf{cl}(A)$ is closed, we have to prove that $\mathsf{cl}(\mathsf{cl}(A)) \subseteq \mathsf{cl}(A)$.

Proposition 1.18. *Let (X, d) be a metric space. For all $A \subseteq X$, $\mathsf{int}(A)$ is open and $\mathsf{cl}(A)$ is closed.*

Proof. Let $A \subseteq X$ be arbitrary.

- We first show that $\mathsf{int}(A)$ is open by proving that $\mathsf{int}(A) \subseteq \mathsf{int}(\mathsf{int}(A))$. Let $x \in \mathsf{int}(A)$ be arbitrary. We need to show that x is an interior point of $\mathsf{int}(A)$. Since x is an interior point of A, we can fix $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. Using Proposition 1.12, we then have $\mathsf{int}(B_\varepsilon(x)) \subseteq \mathsf{int}(A)$. We know that $\mathsf{int}(B_\varepsilon(x)) = B_\varepsilon(x)$ from Proposition 1.17, and hence $B_\varepsilon(x) \subseteq \mathsf{int}(A)$. Since $\varepsilon > 0$, we conclude that x is an interior point of $\mathsf{int}(A)$, which completes the proof.

- We now show that $\mathsf{cl}(A)$ is closed by proving that $\mathsf{cl}(\mathsf{cl}(A)) \subseteq \mathsf{cl}(A)$. Let $x \in \mathsf{cl}(\mathsf{cl}(A))$ be arbitrary, so x is a closure point of $\mathsf{cl}(A)$. We need to show that x is a closure point of A. Let $\varepsilon > 0$ be arbitrary. Since x is a closure point of $\mathsf{cl}(A)$, we know know that $B_{\varepsilon/2}(b) \cap \mathsf{cl}(A) \neq \emptyset$, so we fix a $y \in B_{\varepsilon/2}(x) \cap \mathsf{cl}(A)$. Since $y \in \mathsf{cl}(A)$, we know that y is a closure point of A, so $B_{\varepsilon/2}(y) \cap A \neq \emptyset$, and hence we can fix a $z \in B_{\varepsilon/2}(y) \cap A$. We then have

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{(since } y \in B_{\varepsilon/2}(x) \text{ and } z \in B_{\varepsilon/2}(y)\text{)} \\ &= \varepsilon, \end{aligned}$$

  so $z \in B_\varepsilon(x)$. Since we also have $z \in A$, we conclude that $B_\varepsilon(x) \cap A \neq \emptyset$. As $\varepsilon > 0$ was arbitrary, it follows that x is a closure point of A, i.e. that $x \in \mathsf{cl}(A)$, completing the proof.

□

There is a strong complementary relationship between open and closed sets. For a simple example, an interior point is one that satisfies a "there exists" statement, while a closure point is one that satisfies a "for all" statement. More interestingly, in the proof of Proposition 1.16, we saw that the union proof for open sets was easier than the intersection proof, while the intersection proof for closed sets was easier than the union proof. We can formalize these vague ideas in the following important result.

Proposition 1.19. *Let (X, d) be a metric space and let $A \subseteq X$. We have that A is open if and only if $A^c = X \backslash A$ is closed.*

Proof. Suppose first that A is open. We argue that A^c is closed. Let $x \in \mathsf{cl}(A^c)$ be arbitrary, so $B_\delta(x) \cap A^c \neq \emptyset$ for every $\delta > 0$. If $x \notin A^c$, then $x \in A$, so we can fix $\delta > 0$ with $B_\delta(x) \subseteq A$ (since A is open), which implies that $B_\delta(x) \cap A^c = \emptyset$, a contradiction. Therefore, we must have $x \in A^c$. It follows that A^c is closed.

Suppose conversely that A^c is closed. We argue that A is open. Let $x \in A$ be arbitrary. Since A^c is closed and $x \notin A^c$, we can fix $\delta > 0$ such that $B_\delta(x) \cap A^c = \emptyset$. We then have $B_\delta(x) \subseteq A$, so x is an interior point of A. Therefore, A is open. □

We can use this result together with a few simple set-theoretic facts to save work. For example, suppose that we have proved that whenever $A_1, A_2, \dots, A_n$ are open sets, we have that $A_1 \cup A_2 \cup \cdots \cup A_n$ is open (i.e. the very first part of Proposition 1.16). We can use this fact together with Proposition 1.19 to argue that whenever $A_1, A_2, \dots, A_n$ are closed sets, we have that $A_1 \cap A_2 \cap \cdots \cap A_n$ is closed (i.e. the very last part of Proposition 1.16). To see this, let $A_1, A_2, \dots, A_n$ be arbitrary closed sets. We then have that each A_i^c is both open by Proposition 1.19, so we can conclude that $A_1^c \cup A_2^c \cup \cdots \cup A_n^c$ is open. Using the fact that

$$A_1^c \cup A_2^c \cup \cdots \cup A_n^c = (A_1 \cap A_2 \cap \cdots \cap A_n)^c,$$

it follows that $(A_1 \cap A_2 \cap \cdots \cap A_n)^c$ is open. Since $A_1 \cap A_2 \cap \cdots \cap A_n = ((A_1 \cap A_2 \cap \cdots \cap A_n)^c)^c$, we can use Proposition 1.19 again to conclude that $A_1 \cap A_2 \cap \cdots \cap A_n$ is closed.

Although finite unions and intersections are interesting and fundamental, we can also consider infinite unions and intersections. Given a sequence $A_1, A_2, A_3, \dots$ of sets, we define

$$\bigcup_{n=1}^{\infty} A_n = \{x : \text{There exists } n \in \mathbb{N}^+ \text{ with } x \in A_n\},$$
$$\bigcap_{n=1}^{\infty} A_n = \{x : \text{For all } n \in \mathbb{N}^+, \text{ we have } x \in A_n\}.$$

In this case, we have a family of sets that are indexed by positive natural numbers. Can we do the same with other "index sets"? What if we had a set A_r for each $r \in \mathbb{R}$? After a little thought, we see that there is nothing special about using $\mathbb{N}^+$ above. In fact, given *any* set I, if we have a set A_i for each $i \in I$, then we can still talk about the "general union" and "general intersection". When we use a set I to index sets in this way, we naturally call I an *index set*. If we have such a set I, together with sets A_i for each $i \in I$, then we define

$$\bigcup_{i \in I} A_i = \{x : \text{There exists } i \in I \text{ with } x \in A_i\}$$
$$\bigcap_{i \in I} A_i = \{x : \text{For all } i \in I, \text{ we have } x \in A_i\}$$

We now ask whether the general union, or general intersection, of a family of open (resp. closed) sets is still open (resp. closed). Unfortunately, the answer is no, even for families indexed by $\mathbb{N}^+$. For example, working in $\mathbb{R}$ with the usual metric, the open interval $(-\frac{1}{n}, \frac{1}{n})$ is open for each $n \in \mathbb{N}^+$, but

$$\bigcap_{n \in \mathbb{N}^+} (-1/n, 1/n) = \{0\},$$

which is not open. We can even get sets that are neither open nor closed in this way. For example, we have

$$\bigcap_{n \in \mathbb{N}^+} (0, 1 + 1/n) = (0, 1].$$

Similarly, the general union of closed sets need not be closed, even in the case where we index over $\mathbb{N}^+$. For example, we have

$$\bigcup_{n \in \mathbb{N}^+} [1/n, 3 - (1/n)] = (0, 3).$$

For a more interesting example, if we use $\mathbb{Q}$ as our index set, and let $A_q = \{q\}$ for all $q \in \mathbb{Q}$, then each A_q is closed, but

$$\bigcup_{q \in \mathbb{Q}} \{q\} = \mathbb{Q},$$

which is neither open nor closed. Fortunately, we do have the following result which says that open sets behave well under *arbitrary* unions, and closed sets behave well under *arbitrary* intersections. The proof simply follows the outline of the corresponding arguments in the finite case (Proposition 1.20). It is instructive (and strongly encouraged!) to look at the other two arguments in that proof to see why they do *not* adapt to the infinite case.

Proposition 1.20. *Let (X, d) be a metric space, let I an index set, and suppose we have sets A_i for each $i \in I$.*

1. *If A_i is open for all $i \in I$, then $\bigcup_{i \in I} A_i$ is open.*
2. *If A_i is closed for all $i \in I$, then $\bigcap_{i \in I} A_i$ is closed.*

Proof. Assume that A_i is open for all $i \in I$. Let $x \in \bigcup_{i \in I} A_i$ be arbitrary. By definition, we can then fix $j \in I$ with $x \in A_j$. Since A_j is open, we know that $x \mathsf{int}(A_j)$, so we can fix $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_j$. Since $A_j \subseteq \bigcup_{i \in I} A_i$, it follows that $B_\varepsilon(x) \subseteq \bigcup_{i \in I} A_i$. Therefore, $x \in \mathsf{int}(\bigcup_{i \in I} A_i)$.

Assume now that A_i is closed for all i. Let $x \in \mathsf{cl}(\bigcap_{i \in I} A_i)$ be arbitrary. We first show that $x \in \mathsf{cl}(A_i)$ for all i. Let $\varepsilon > 0$ be arbitrary. Since $x \in \mathsf{cl}(\bigcap_{i \in I} A_i)$, we know that $B_\varepsilon(x) \cap (\bigcap_{i \in I} A_i) \neq \emptyset$. Now $B_\varepsilon(x) \cap (\bigcap_{i \in I} A_i)$ is a subset $B_\varepsilon(x) \cap A_i$ for all i, so $B_\varepsilon(x) \cap A_i \neq \emptyset$ for all i. Since $\varepsilon > 0$ was arbitrary, we conclude that $x \in \mathsf{cl}(A_i)$ for all i. Recall that each A_i is closed, so it follows that $x \in A_i$ for all i, and hence $x \in \bigcap_{i \in I} A_i$. □

We can use these ideas to construct an example of a very interesting closed set in $\mathbb{R}$ that has some counterintuitive properties.

Definition 1.21. *We define a sequence of sets recursively. We start by letting $C_0 = [0, 1]$. Suppose that C_n is a pairwise disjoint union of 2^n many closed intervals, each of length $\frac{1}{3^n}$. We then let C_{n+1} be the result of removing the open intervals that are the middle third of each interval in C_n, so C_{n+1} is a pairwise disjoint union of 2^{n+1} many closed intervals, each of length $\frac{1}{3^{n+1}}$. This completes the recursive definition of the sets C_n. We then define*

$$C = \bigcap_{n=0}^{\infty} C_n.$$

The set C is called the Cantor set.

Notice that each C_n is closed (because it is a finite union of closed intervals), and hence C is closed by Proposition 1.20. The set C has many interesting properties. If one tries to determine how "big" C is, one approach would be to determine how much of $[0, 1]$ we have removed at each stage. Notice that we removed an interval of length $\frac{1}{3}$ when constructing C_1, then removed 2 intervals of length $\frac{1}{9}$ when constructing C_2, then removed $4 = 2^2$ intervals of length $(\frac{1}{3})^3$ when constructing C_3, etc. If we follow this logic, the we can determine the total amount removed by looking at the infinite series

$$\frac{1}{3} + 2 \cdot \left(\frac{1}{3}\right)^2 + 2^2 \cdot \left(\frac{1}{3}\right)^3 + 2^3 \cdot \left(\frac{1}{3}\right)^4 + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^n.$$

Since this is a geometric series and $|\frac{2}{3}| < 1$, we know that it converges and that

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^n = \frac{1/3}{1-(2/3)} = \frac{1/3}{1/3} = 1.$$

In other words, it appears that we have removed the full length of $[0,1]$. However, this does not mean that $C = \emptyset$. In fact, we certainly have $0, 1 \in C$. Moreover, one can prove inductively that the endpoints of any C_n are endpoints of C_m for all $m \geq n$, and hence C contains the endpoints of each of the intervals in C_n. Thus, C is infinite. More surprisingly, C is uncountable!

1.4 Compact Sets

Definition 1.22. *Let (X, d) be a metric space and let $A \subseteq X$. An* open cover *of A is a collection of open sets $\{D_i : i \in I\}$, where I is some index set, such that $A \subseteq \bigcup_{i \in I} D_i$.*

Working in $\mathbb{R}$ (with the usual metric), each open interval $(-n, n)$ for $n \in \mathbb{N}^+$ is an open set. Thus, we can consider the collection $\{(-n,n) : n \in \mathbb{N}^+\}$ of all of these open sets. In this case, our index set is $I = \mathbb{N}^+$, and we are letting $D_i = (-i, i)$ for each $i \in I$. Notice that for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $|x| < n$, from which it follows that $x \in (-n, n)$. We have $\mathbb{R} \subseteq \bigcup_{n \in \mathbb{N}^+} (-n, n)$, so $\{(-n,n) : n \in \mathbb{N}^+\}$ is an open cover of $\mathbb{R}$. In fact, given any $A \subseteq \mathbb{R}$, the collection $\{(-n,n) : n \in \mathbb{N}^+\}$ is an open cover of A because $A \subseteq \mathbb{R} \subseteq \bigcup_{n \in \mathbb{N}^+} (-n, n)$

For another example, consider the collection of open sets $\{(1/n, 1) : n \in \mathbb{N}^+\}$. Here, we are again using $I = \mathbb{N}^+$, but now we are letting $D_i = (1/i, 1)$ for each $i \in \mathbb{N}^+$ (and using $D_1 = (1,1) = \emptyset$, which is also open). It is straightforward to check that $\{(1/n,1) : n \in \mathbb{N}^+\}$ is an open cover of $(0,1)$. As above, given any $A \subseteq (0,1)$, the collection $\{(1/n,1) : n \in \mathbb{N}^+\}$ is also an open cover of A. For example, $\{(1/n,1) : n \in \mathbb{N}^+\}$ is an open cover of $[\frac{1}{4}, 1)$. However, notice that $\{(1/n,1) : n \in \mathbb{N}^+\}$ is *not* an open cover of $[0,1)$, because $0 \notin (1/n, 1)$ for any $n \in \mathbb{N}^+$.

Definition 1.23. *Let (X, d) be a metric space and let $A \subseteq X$. We say that A is* compact *if every open cover of A has a finite subcover, i.e. whenever $\{D_i : i \in I\}$ is an open cover of A, there exists a finite set $F \subseteq I$ such that $A \subseteq \bigcup_{i \in F} D_i$.*

The definition of compact may be the least intuitive definition that you have seen in mathematics up to this point. It is certainly abstract, and it takes some time to develop an appreciation for its elegance and power. We start by giving a couple of examples of sets that are *not* compact in $\mathbb{R}$ (under the usual metric):

- The set $\mathbb{R}$ itself is not compact: As we saw above, the collection of sets $\{(-n,n) : n \in \mathbb{N}^+\}$ is an open cover of $\mathbb{R}$. However, there is no finite $F \subseteq \mathbb{N}^+$ with $\mathbb{R} \subseteq \bigcup_{n \in F} (-n, n)$. To see this, suppose that $F \subseteq \mathbb{N}^+$ is finite. If $F = \emptyset$, then $\bigcup_{n \in F} (-n, n) = \emptyset$, which is certainly not an open cover of $\mathbb{R}$. Suppose then that $F \neq \emptyset$, and let $N = \max(F)$. We then have $(-n,n) \subseteq (-N, N)$ for all $n \in F$, so $\bigcup_{n \in F} (-n, n) = (-N, N)$, and hence $\{(-n,n) : n \in F\}$ is not an open cover of $\mathbb{R}$. We have shown that the open cover $\{(-n,n) : n \in \mathbb{N}^+\}$ of $\mathbb{R}$ does not have a finite subcover, so $\mathbb{R}$ is not compact.

- The set $(0,1)$ is not compact: As we saw above, the collection of sets $\{(1/n,1) : n \in \mathbb{N}^+\}$ is an open cover of $(0,1)$. However, there is no finite $F \subseteq \mathbb{N}^+$ with $(0,1) \subseteq \bigcup_{n \in F} (1/n, 1)$. To see this, suppose that $F \subseteq \mathbb{N}^+$ is finite. We may assume that $F \neq \emptyset$ as above, and let $N = \max(F)$. We then have $(1/n, 1) \subseteq (1/N, 1)$ for all $n \in F$, so $\bigcup_{n \in F} (1/n, 1) = (1/N, 1)$, and hence $\{(1/n,1) : n \in F\}$ is not an open cover of $(0,1)$. We have shown that the open cover $\{(1/n,1) : n \in \mathbb{N}^+\}$ of $(0,1)$ does not have a finite subcover, so $(0,1)$ is not compact.

Consider the half-open interval $[\frac{1}{4}, 1) \subseteq \mathbb{R}$. As mentioned above, the collection of sets $\{(1/n,1) : n \in \mathbb{N}^+\}$ is an open cover of $[\frac{1}{4}, 1)$. Now this particular open cover of $[\frac{1}{4}, 1)$ does have a finite subcover. For example, letting $F = \{5\}$, we have $\bigcup_{n \in F}(1/n, 1) = (1/5, 1)$, which is a finite subcover of $[\frac{1}{4}, 1)$. There are also many other finite subsets of $\mathbb{N}^+$ that also work, because any finite $F \subseteq \mathbb{N}^+$ which contains an element greater than 4 will satisfy $[1/4, 1) \subseteq \bigcup_{n \in F}(1/n, 1)$. However, it is important to note that we can *not* conclude from this one collection that $[1/4, 1)$ is compact, as the definition of compact requires that *every* open cover has a finite subcover. In this case, the collection $\{(0, 1 - 1/n) : n \in \mathbb{N}^+\}$ is an open cover of $[1/4, 1)$ that does not have a finite subcover, so in fact $[1/4, 1)$ is not compact.

With this background, it might seem very difficult to prove that a given set $A \subseteq \mathbb{R}$ is compact. In general, this is certainly true, but it is relatively straightforward to handle finite sets.

Proposition 1.24. *Let (X, d) be a metric space. Every finite subset of X is compact.*

Proof. Let $A \subseteq X$ be finite. If $A = \emptyset$, then A is trivially compact because given any open cover $\{D_i : i \in I\}$ of $\emptyset$, we can let $F = \emptyset$. Suppose then that $A \neq \emptyset$, and write $A = \{a_1, a_2, \dots, a_n\}$. Let $\{D_i : i \in I\}$ be an arbitrary open cover of A. For each k with $1 \leq k \leq n$, we have $a_k \in \bigcup_{i \in I} D_i$, so we can fix $i_k \in I$ with $a_k \in D_{i_k}$. Let $F = \{i_1, i_2, \dots, i_n\}$. We then have that F is finite and $A \subseteq \bigcup_{i \in F} D_i$, so the open cover $\{D_i : i \in I\}$ of A has a finite subcover. $\square$

Perhaps surprisingly, there do exist infinite compact sets. Before trying to come up with an example, we first prove the following result, which restricts the potential options. The proof is a generalization of the important examples we discussed above.

Proposition 1.25. *Let (X, d) be a metric space. Every compact subset of X is both closed and bounded.*

Proof. Let $A \subseteq X$ be a compact set. If $A = \emptyset$, then it is trivially closed and bounded, so assume that $A \neq \emptyset$.

- We first prove that A is bounded. Since $A \neq \emptyset$, we can fix $a \in A$. Consider the collection of sets $\{B_r(a) : r \in \mathbb{N}^+\}$. We have $X = \bigcup_{n \in \mathbb{N}^+} B_r(a)$, so we know that $\{B_r(a) : r \in \mathbb{N}^+\}$ is an open cover of A. As A is compact, we can fix a finite set $F \subseteq \mathbb{N}^+$ with $A \subseteq \bigcup_{i \in F} B_r(a)$. Since $A \neq \emptyset$, we know that $F \neq \emptyset$, so let $m = \max(F)$. We then have $A \subseteq B_m(a)$, so A is bounded (see the homework).

- We next prove that A is closed, which we do by showing that the complement A^c is open. Let $x \in A^c$ be arbitrary. We show that $x \in \mathsf{int}(A^c)$. For each $n \in \mathbb{N}^+$, let
$$D_n = \{y \in X : d(x, y) > 1/n\}.$$
Since each $\{y \in X : d(x, y) \leq 1/n\}$ is closed (by the homework) and
$$D_n = X \backslash \{y \in X : d(x, y) \leq 1/n\},$$
we can use Proposition 1.19 to conclude that each D_n is open. Notice that
$$\bigcup_{n \in \mathbb{N}^+} D_n = X \backslash \{x\}.$$
Since $x \notin A$, we have
$$A \subseteq \bigcup_{n \in \mathbb{N}^+} D_n,$$
and hence $\{D_n : n \in \mathbb{N}^+\}$ is an open cover of A. Using the fact that A is compact, we can fix a finite set $F \subseteq \mathbb{N}^+$ such that $A \subseteq \bigcup_{n \in F} D_n$. Since $A \neq \emptyset$, we know that $F \neq \emptyset$, so let $N = \max(F)$. Since $D_m \subseteq D_n$ whenever $m \leq n$, it follows that $\bigcup_{n \in F} D_n = D_N$. Therefore, we have $A \subseteq D_N$, i.e.
$$A \subseteq \{y \in X : d(x, y) > 1/N\}.$$
It follows that
$$\{y \in X : d(x, y) < 1/N\} \subseteq A^c,$$
which is to say that $B_{1/N}(x) \subseteq A^c$. Hence $x \in \mathsf{int}(A^c)$. We have shown that every element of A^c is an element of $\mathsf{int}(A^c)$, so A^c is open. Using Proposition 1.19, we conclude that $A = (A^c)^c$ is closed.

□

Our next result tells us that if we happen to stumble across a "big" compact set in metric space, then we will be able to obtain many other compact sets inside of it.

Proposition 1.26. *Let (X, d) be a metric space. Every closed subset of a compact set is compact. In other words, if $A \subseteq X$ is compact, and $B \subseteq A$ is closed, then B is compact.*

Proof. Suppose that $A \subseteq X$ is compact and $B \subseteq A$ is closed. Let $\{D_i : i \in I\}$ be an arbitrary open cover of B. Since B is closed, we know from Proposition 1.19 that B^c is open. Notice that $\{B^c\} \cup \{D_i : i \in I\}$ is an open cover of X, and so it is certainly an open cover of A. Since A is compact, we can fix a finite set $F \subseteq I$ such that $\{B^c\} \cup \{D_i : i \in F\}$ is an open cover of A. As $B \subseteq A$, we know that $\{B^c\} \cup \{D_i : i \in F\}$ is an open cover of B. But $B \cap B^c = \emptyset$, so $\{D_i : i \in F\}$ is a finite subcover of B. □

For the rest of this section, we will turn our attention to $\mathbb{R}^n$ (with the usual metric), as the situation is more intricate in general metric spaces. We start by examining $\mathbb{R}$. If we are looking for an example of an infinite compact subset of $\mathbb{R}$, then Proposition 1.25 tells us that we must look to closed and bounded sets. The simplest example of an infinite closed and bounded set is a closed interval. We now prove that *every* closed interval is compact.

Proposition 1.27. *Let $c, d \in \mathbb{R}$ with $c < d$. We then have that $[c, d]$ is a compact subset of $\mathbb{R}$.*

Proof. Let $\{D_i : i \in I\}$ be an arbitrary open cover of $[c, d]$. Let

$$B = \{x \in [c, d] : \text{There exists a finite set } F \subseteq I \text{ with } [c, x] \subseteq \bigcup_{i \in F} D_i\}.$$

Notice that $c \in B$ trivially because c is an element of some D_i, and we can fix such an i and let $F = \{i\}$. Also, B is bounded above by d. Thus, we can let $s = \sup B$. Since d is an upper bound of B, it follows that $s \leq d$. Also, since $c \in B$, we have $c \leq s$.

We first claim that $s = d$. Suppose instead that $s < d$. Since $c \leq s$, we have $s \in [c, d]$, so we can fix a $j \in I$ with $s \in D_j$. Since D_j is open, we can fix $\varepsilon > 0$ with $V_\varepsilon(s) \subseteq D_j$. Now $s - \varepsilon < s$, so $s - \varepsilon$ is not an upper bound of B, and hence we can fix $x \in B$ with $x > s - \varepsilon$. By definition of B, we can fix a finite set $F \subseteq I$ with $[c, x] \subseteq \bigcup_{i \in F} D_i$. Letting $G = F \cup \{j\}$ and $\delta = \min\{d - c, \frac{\varepsilon}{2}\} > 0$, we then have that G is finite and $s < s + \delta \leq d$. Furthermore, $[c, s + \delta] \subseteq \bigcup_{i \in G} D_i$, so $s + \delta \in B$, contradicting the fact that s is an upper bound of B. Therefore, we must have $s = d$.

We now show that the open cover $\{D_i : i \in I\}$ of $[c, d]$ has a finite subcover. Since $\{D_i : i \in I\}$ is an open cover of $[c, d]$, we can fix $j \in I$ with $d \in D_j$. Since D_j is open, we can fix $\varepsilon > 0$ with $V_\varepsilon(s) \subseteq D_j$. As $d - \varepsilon < d$, we know that $d - \varepsilon$ is not an upper bound of B, and hence we can fix $x \in B$ with $x > d - \varepsilon$. By definition of B, we can fix a finite set $F \subseteq I$ with $[c, x] \subseteq \bigcup_{i \in F} D_i$. Letting $G = F \cup \{j\}$, we then have that G is finite and that $[c, d] \subseteq \bigcup_{i \in G} D_i$. Thus, we have found a finite subcover of $[c, d]$. □

Our next goal is to lift this result to $\mathbb{R}^n$. The natural analogue of a closed and bounded interval in $\mathbb{R}$, is a rectangle $[c_1, d_1] \times [c_2, d_2]$ in $\mathbb{R}^2$, a rectangular prism $[c_1, d_1] \times [c_2, d_2] \times [c_3, d_3]$ in $\mathbb{R}^3$, etc. In order to ladder up to these higher dimensions using Proposition 1.27, we will prove a general result about the product of two compact subsets of Euclidean space. We first need the following lemma.

Lemma 1.28. *Suppose that $D \subseteq \mathbb{R}^{m+n}$ is open, and that $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \in D$. There exists open sets $U \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^n$ such that $(x_1, \ldots, x_n) \in U$, $(x_{m+1}, \ldots, x_{m+n}) \in W$, and $U \times W \subseteq D$.*

Proof. Since D is open and $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \in D$, we can fix $\varepsilon > 0$ with $B_\varepsilon(x_1, \ldots, x_n, x_{n+1}) \subseteq D$. Let $U = B_{\varepsilon/\sqrt{2}}(x_1, \ldots, x_m) \subseteq \mathbb{R}^m$ and let $W = B_{\varepsilon/\sqrt{2}}(x_{m+1}, \ldots, x_{m+n}) \subseteq \mathbb{R}^n$ and note that U and W

are open. For any $(y_1, \dots, y_m, y_{m+1}, \dots, y_{m+n}) \in U \times W$, we have $(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2 \leq \frac{\varepsilon^2}{2}$ and $(x_{m+1} - y_{m+1})^2 + \cdots + (x_{m+n} - y_{m+n})^2 < \frac{\varepsilon^2}{2}$, so

$$\begin{aligned}\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 + (x_{n+1} - y_{n+1})^2 + \cdots + (x_{m+n} - y_{m+n})^2} &\leq \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} \\ &= \sqrt{\varepsilon^2} \\ &= \varepsilon,\end{aligned}$$

and hence $(y_1, \dots, y_n, y_{m+1}, \dots, y_{m+n}) \in B_\varepsilon(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n}) \subseteq D$. □

Proposition 1.29. *Let $A \subseteq \mathbb{R}^m$ and $C \subseteq \mathbb{R}^n$ both be compact. We then have that $A \times C \subseteq \mathbb{R}^{m+n}$ is compact.*

Proof. Let $\{D_i : i \in I\}$ be an arbitrary open cover of $A \times C$. Each $(a, c) \in A \times C$ is an element some D_i, so by Lemma 1.28 we can fix open sets $U_{(a,c)} \subseteq \mathbb{R}^m$ and $W_{(a,c)} \subseteq \mathbb{R}^n$ with $(a, c) \in U_{(a,c)} \times W_{(a,c)}$ and where $U_{(a,c)} \times W_{(a,c)}$ contained in some D_i. It suffices to find a finite subcover of $A \times C$ from the set $\{U_{(a,c)} \times W_{(a,c)} : (a, c) \in A \times C\}$, because we can take one such corresponding $i \in I$ for each (a, c) in our finite set in order to obtain a finite subcover of $\{D_i : i \in I\}$.

Now fix an element $a \in A$, and consider the corresponding cross section. The set $\{W_{(a,c)} : c \in C\}$ is an open cover of the compact set C, so we can fix a finite $G_a \subseteq C$ such that $\{W_{(a,c)} : c \in G_a\}$ covers C. For each $a \in A$, let $U_a = \bigcap_{c \in G_a} U_{(a,c)}$, and note that $U_a \subseteq \mathbb{R}^n$ is open by Proposition 1.16. Now $\{U_a : a \in A\}$ is an open cover of the compact set A, so we can fix a finite $F \subseteq A$ such that $\{U_a : a \in F\}$ covers A. The set $\{U_a \times W_{(a,c)} : a \in F, c \in G_a\}$ covers $A \times C$, and hence $\{U_{(a,c)} \times W_{(a,c)} : a \in F, c \in G_a\}$ covers $A \times C$, so we have found a finite subcover of $A \times C$ from the set $\{U_{(a,c)} \times W_{(a,c)} : (a, c) \in A \times C\}$. □

Corollary 1.30. *Let $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{R}$ with $c_i < d_i$ for all i. We then have that $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is a compact subset of $\mathbb{R}^n$.*

Proof. By induction using Proposition 1.27 and Proposition 1.29. □

We have now down all of the hard work to give a few other characterizations of compact sets in $\mathbb{R}^n$. I should note that (1) and (3) are equivalent in any metric space, although the proof is somewhat more complicated there. However, in a general metric space, it is absolutely possible for to have closed and bounded sets that are *not* compact. The replacement for (2) in this more general setting is more complicated.

Theorem 1.31 (Heine-Borel)**.** *Let $A \subseteq \mathbb{R}^n$. The following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded.*
3. *Every sequence $\langle a_k \rangle$ from A (i.e. where $a_k \in A$ for all $k \in \mathbb{N}^+$) has a subsequence that converges to an element of A.*

Proof.
- (1) $\Rightarrow$ (2): Immediate from Proposition 1.25.
- (2) $\Rightarrow$ (1): Suppose that A is closed is bounded. Since A is bounded, we can fix an $M \in \mathbb{R}$ with $A \subseteq [-M, M]^n$. Using Corollary 1.30, we know that $[-M, M]^n$ is compact, so as A is closed, we can use Proposition 1.26 to conclude that A is compact.
- (2) $\Rightarrow$ (3): Suppose that A is closed and bounded, and let $\langle a_k \rangle_{k \in \mathbb{N}^+}$ be a sequence from A. For each k, let $a_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$ where $x_{k,i} \in \mathbb{R}$. The sequence $\langle x_{k,1} \rangle_{k \in \mathbb{N}^+}$ is a bounded sequence in $\mathbb{R}$, so we can extract a convergent subsequence of it by the Bolzano-Weierstrass Theorem. Let I_1 be the corresponding infinite set of indices of this sequence, and let b_1 be the limit of this subsequence. Now

$\langle x_{k,2}\rangle_{k\in I_1}$ is a bounded sequence in $\mathbb{R}$, so we can extract a convergent subsequence of it by the Bolzano-Weierstrass Theorem. Let $I_2 \subseteq I_1$ be the corresponding infinite set of indices of this sequence, and let b_1 be the limit of this subsequence. Continue in this way, repeatedly thinning out the infinite set of indices, until we arrive at I_n together with corresponding limit b_n. Since $\langle x_{k,1}\rangle_{k\in I_n}$ is a subsequence of $\langle x_{k,1}\rangle_{k\in I_1}$, it also converges to b_1. In general, for each i, the sequence $\langle x_{k,i}\rangle_{k\in I_n}$ converges to b_i. Using (the generalization of) Problem 4 on Homework 1, the subsequence $\langle a_k\rangle_{k\in I_n}$ of $\langle a_k\rangle_{k\in\mathbb{N}^+}$ converges to $(b_1, b_2, \dots, b_n)$. Since $a_k \in A$ for all $k \in I_n$, we know from Proposition 1.13 that $(b_1, b_2, \dots, b_n) \in \mathsf{cl}(A)$. Using the fact that A is closed, we conclude that $(b_1, b_2, \dots, b_n) \in A$.

- (3) $\Rightarrow$ (2): We prove the contrapositive, i.e. that if A is either not closed or not bounded, then there is a sequence $\langle a_k\rangle$ from A such that no subsequence of $\langle a_k\rangle$ converges to a point of A.
  - Suppose first that A is not closed. We then have that $\mathsf{cl}(A) \neq A$. Since we know that $A \subseteq \mathrm{cl}(A)$ from Proposition 1.11, we can fix $b \in \mathsf{cl}(A)\backslash A$. As $b \in \mathsf{cl}(A)$, we can use Proposition 1.13 to fix a sequence $\langle a_k\rangle$ from A that converges to b. Now every subsequence of $\langle a_k\rangle$ will also converge to b (a good exercise), so in particular no subsequence of $\langle a_k\rangle$ converges to a point of A.
  - Suppose that A is not bounded. Define a sequence $\langle a_k\rangle$ as follows. Given $k \in \mathbb{N}^+$, we know that $A \not\subseteq B_k(\vec{0})$ because A is not bounded, so can let a_k be some element of A with $d(\vec{0}, a_k) \geq k$. Now every subsequence of $\langle a_k\rangle$ will also fail to be bounded (another good exercise is to check that convergent sequences are bounded), and hence every subsequence of A will fail to converge. In particular, no subsequence of $\langle a_k\rangle$ converges to a point of A.

□

1.5 Continuous Functions

We first generalize the definition of continuous to metric spaces in the natural way.

Definition 1.32. *Let (X, d_1) and (Y, d_2) be metric spaces, let $f : X \to Y$, and let $a \in X$. We say that f is* continuous at a *if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$ with $d_1(x, a) < \delta$, we have $d_2(f(x), f(a)) < \varepsilon$.*

Definition 1.33. *Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$. We say that f is* continuous *if it is continuous at every $a \in X$.*

Our first task is give an incredibly powerful characterization of continuous functions in terms of preimages of sets in the codomain. We start with the following notation, which is truly terrible but entrenched and standard.

Definition 1.34. *Let $f : A \to B$ be a function.*

- *For any $C \subseteq A$, we define $f(C) = \{f(c) : c \in C\}$.*
- *For any $D \subseteq B$, we define $f^{-1}(D) = \{a \in A : f(a) \in D\}$.*

For our purposes, we will find that the latter of these two is significantly more important. One of the primary reasons why is that preimages interact well with all of the standard set-theoretic operations. Think about how some of these could fail if worked with images (i.e. the first of these definitions) rather than preimages.

Proposition 1.35. *Let $f : A \to B$ be a function.*

1. *For any set I together with sets $D_i \subseteq B$ for each $i \in I$, we have*

$$f^{-1}\left(\bigcup_{i\in I} D_i\right) = \bigcup_{i\in I} f^{-1}(D_i).$$

2. *For any set I indexing sets $D_i \subseteq B$, we have*

$$f^{-1}\left(\bigcap_{i\in I} D_i\right) = \bigcap_{i\in I} f^{-1}(D_i).$$

3. *For all $D_1, D_2 \subseteq B$, we have $f^{-1}(D_1 \backslash D_2) = f^{-1}(D_1) \backslash f^{-1}(D_2)$.*

4. *For all $D \subseteq B$, we have $f^{-1}(B \backslash D) = A \backslash f^{-1}(D)$.*

5. *For all $D \subseteq B$, we have $f(f^{-1}(D)) \subseteq D$.*

Proof. Exercise. □

Our main result says that a function between metric spaces is continuous exactly when the preimage of every open set is open.

Proposition 1.36. *Let (X, d_1) and (Y, d_2) be metric spaces. A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(D)$ is open in X whenever D is open in Y.*

Proof. Suppose first that $f\colon X \to Y$ is continuous. Let D be an arbitrary open subset of X. We show that $f^{-1}(D)$ is open. Let $w \in f^{-1}(D)$ be arbitrary. By definition, we then have $f(w) \in D$, so as D is open, we can fix $\varepsilon > 0$ with $B_\varepsilon(f(w)) \subseteq D$. Since f is continuous at w, we can fix $\delta > 0$ such that for all $x \in X$ with $d_1(x, w) < \delta$, we have $d_2(f(x), f(w)) < \varepsilon$. We claim that $B_\delta(x) \subseteq f^{-1}(D)$. To see this, notice that given any $x \in B_\delta(w)$, we have $d_1(x, w) < \delta$, so $d_2(f(x), f(w)) < \varepsilon$, hence $f(x) \in B_\varepsilon(f(w)) \subseteq D$, and therefore $x \in f^{-1}(D)$. We have shown that for any $w \in f^{-1}(D)$, there exists $\delta > 0$ such that $B_\delta(w) \subseteq f^{-1}(D)$. Thus, $f^{-1}(D)$ is open.

Suppose conversely that $f\colon X \to Y$ has the property that $f^{-1}(D)$ is open in X whenever D is open in Y. We show that f is continuous. Let $a \in X$ be arbitrary. We show that f is continuous at a. Let $\varepsilon > 0$ be arbitrary. Notice that $B_\varepsilon(f(a))$ is open in Y by Proposition 1.17, so $f^{-1}(B_\varepsilon(f(a)))$ is open in X by assumption. Now we trivially have $f(a) \in B_\varepsilon(f(a))$, so $a \in f^{-1}(B_\varepsilon(f(a)))$. Combining this with the fact that $f^{-1}(B_\varepsilon(f(a)))$ is open in X, we can fix $\delta > 0$ so that $B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a)))$. Now given any $x \in X$ with $d_1(x, a) < \delta$, we have $x \in B_\delta(a)$, so $x \in f^{-1}(B_\varepsilon(f(a))$, hence $f(x) \in B_\varepsilon(f(a))$, and therefore $d_2(f(x), f(a)) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that f is continuous at a. As $a \in X$ was arbitrary, it follows that f is continuous. □

Continuity also interacts well with compact sets, although in this case we deal with an image rather than a preimage. The result is usually summarized as saying that "the continuous image of a compact set is compact".

Proposition 1.37. *Let (X, d_1) and (Y, d_2) be metric spaces, let $f\colon X \to Y$ be continuous, and let $A \subseteq X$ be compact. We then have that $f(A) = \{f(a) : a \in A\}$ is compact.*

Proof. Let $\{D_i : i \in I\}$ be an arbitrary open cover of $f(A)$. By Proposition 1.36, we know that each set $f^{-1}(D_i)$ is open in X. Since $f(A) \subseteq \bigcup_{i\in I} D_i$, it follows that $A \subseteq \bigcup_{i\in I} f^{-1}(D_i)$, so $\{f^{-1}(D_i) : i \in I\}$ is an open cover of A. As A is compact, we can fix a finite set $F \subseteq I$ such that $\{f^{-1}(D_i) : i \in F\}$ is an open cover of A. We then have that $\{D_i : i \in F\}$ is a finite subcover of $f(A)$. □

To see the power of these results, we derive a fundamental result (with little effort) that is used throughout Calculus.

Corollary 1.38 (Extreme Value Theorem)**.** *Let $n \in \mathbb{N}^+$, and let $\mathbb{R}^n$ and $\mathbb{R}$ have the usual metrics. If $A \subseteq \mathbb{R}^n$ is closed and bounded, and $f\colon A \to \mathbb{R}$ is continuous, then f achieves a maximum and minimum value on A.*

Proof. Since A is a closed and bounded subset of $\mathbb{R}^n$, we know that A is compact from the Heine-Borel Theorem, so $f(A)$ is compact from Proposition 1.37. By the homework, this compact subset of $\mathbb{R}$ has a maximum and minimum value. $\square$