Homework 9b: Due Friday, December 1

Problem 1: Let V be an inner product space. Let $T\colon V \to V$ be a linear transformation. Show that the following are equivalent:

1. T preserves the inner product: For all $\vec{u}, \vec{w} \in V$, we have $\langle T(\vec{u}), T(\vec{w}) \rangle = \langle \vec{u}, \vec{w} \rangle$.
2. T preserves the norm: For all $\vec{u} \in V$, we have $||T(\vec{u})|| = ||\vec{u}||$.

A linear transformation with either of (and hence both of) these properties is called an *orthogonal* transformation (or a *unitary* transformation in the complex case).

Problem 2: Let V be an inner product space, and let U be a subspace of V. We saw in class that $U^\perp$ was a subspace of V. Show that $U^\perp$ is a *closed* subspace of V, in the sense that $U^\perp$ is a closed set in the underlying metric space.

Problem 3: Let V be a Hilbert space, and let W be a closed subspace of V. In class, we argued that for each $\vec{v} \in V$, there is a unique $\vec{w} \in W$ minimizing the value $||\vec{v} - \vec{w}||$, i.e. there exists a unique $\vec{w} \in W$ such that $||\vec{v} - \vec{w}|| \leq ||\vec{v} - \vec{x}||$ for all $\vec{x} \in W$. Moreover, for this unique $\vec{w}$, we have $\vec{v} - \vec{w} \in W^\perp$. Define $P\colon V \to W$ by letting $P(\vec{v})$ be this unique $\vec{w}$. Show that P is a linear transformation.