Homework 9a: Due Friday, November 17

Problem 1: For each $n \in \mathbb{N}^+$, define $f_n\colon [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Is $\langle f_n \rangle$ a Cauchy sequence in $L^1[0,1]$? Explain.

Problem 2: Let $f\colon [a,b] \to \mathbb{R}$ be a bounded measurable function. Recall that we originally thought about defining $||f||_\infty$ to be $\sup\{|f(x)| : x \in [a,b]\}$. However, unlike our definition for L^p, such a definition would not be stable if we changed the function on a set of measure 0. As a result, we actually define

$$\begin{aligned} ||f||_\infty &= \inf\{c \in \mathbb{R} : m(\{x \in [a,b] : |f(x)| > c\}) = 0\} \\ &= \inf\{c \in \mathbb{R}^+ : m(f^{-1}((-\infty,-c) \cup (c,\infty))) = 0\}. \end{aligned}$$

The quantity on the right is sometimes called the *essential supremum* of f.
a. Show that if $f, g\colon [a,b] \to \mathbb{R}$ are both bounded measurable functions, and if $f = g$ a.e., then $||f||_\infty = ||g||_\infty$.
b. Show that if $f, g\colon [a,b] \to \mathbb{R}$ are both bounded measurable functions, then

$$||f+g||_\infty \leq ||f||_\infty + ||g||_\infty.$$

Problem 3: Let V be an inner product space.
a. Show that $||\vec{v}+\vec{w}||^2 + ||\vec{v}-\vec{w}||^2 = 2 \cdot ||\vec{v}||^2 + 2 \cdot ||\vec{w}||^2$ for all $\vec{v}, \vec{w} \in V$.
b. Show that $||\vec{v}+\vec{w}||^2 - ||\vec{v}-\vec{w}||^2 = 4 \cdot \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.
c. Show that the function $f\colon V \to \mathbb{R}$ defined by $f(\vec{v}) = ||\vec{v}||$ is continuous.
Aside: Part (a) is known as the *parallelogram law*, because it says that the sums of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of the diagonals.