Homework 8b: Due Friday, November 10

Problem 1: Let $(X, \mathcal{S}, \mu)$ be a measure space. On Homework 8a, you showed that the set

$$\mathcal{S}_\mu = \{E \in \mathcal{P}(X) : \text{There exists } A, B \in \mathcal{S} \text{ with } A \subseteq E \subseteq B \text{ and } \mu(B \backslash A) = 0\}$$

was a σ-algebra on X containing $\mathcal{S}$. Define $\overline{\mu}\colon \mathcal{S}_\mu \to [0, \infty]$ as follows. Given $E \in \mathcal{S}_\mu$, fix some $A, B \in \mathcal{S}$ with $A \subseteq E \subseteq B$ and $\mu(B \backslash A) = 0$, and let $\overline{\mu}(E) = \mu(A)$.
a. Show that $\overline{\mu}$ is well-defined.
b. Show that $\overline{\mu}(E) = \mu(E)$ for all $E \in \mathcal{S}$.
c. Show that $(X, \mathcal{S}_\mu, \overline{\mu})$ is a measure space.
d. Show that for all $E \in \mathcal{S}_\mu$ with $\overline{\mu}(E) = 0$, we have $\mathcal{P}(E) \subseteq \mathcal{S}_\mu$.
Note: This shows that $(X, \mathcal{S}_\mu, \overline{\mu})$ is a complete measure space such that $\mathcal{S}_\mu \supseteq \mathcal{S}$ and $\overline{\mu}$ extends μ. The measure space $(X, \mathcal{S}_\mu, \overline{\mu})$ is (shockingly) called the completion of $(X, \mathcal{S}, \mu)$.

Problem 2:
a. Let X and Y be sets, let $\mathcal{S}$ be a σ-algebra on X, and let $f\colon X \to Y$. Show that

$$\mathcal{T} = \{B \in \mathcal{P}(Y) : f^{-1}(B) \in \mathcal{S}\}.$$

is a σ-algebra on Y.
b. Given a metric space (X, d), recall that we defined the Borel σ-algebra of X to be the smallest σ-algebra on X that contains every open set. Suppose that (X_1, d_1) and (X_2, d_2) are both metric spaces, and that $f\colon X_1 \to X_2$ is continuous. Show that $f^{-1}(B)$ is a Borel subset of X_1 for all Borel subsets B of X_2.