Homework 8a: Due Friday, November 3

Problem 1: Let $(X_1, \mathcal{S}_1, \mu_1)$ and $(X_2, \mathcal{S}_2, \mu_2)$ be two measure spaces, and assume that $X_1 \cap X_2 = \emptyset$ (otherwise, one can simply rename the elements to make this work). Let $Y = X_1 \cup X_2$ and let

$$\mathcal{T} = \{B \in \mathcal{P}(X_1 \cup X_2) : B \cap X_1 \in \mathcal{S}_1 \text{ and } B \cap X_2 \in \mathcal{S}_2\}.$$

Finally, define $\nu \colon \mathcal{T} \to [0, \infty]$ by letting $\nu(B) = \mu_1(B \cap X_1) + \mu_2(B \cap X_2)$. Show that $\mathcal{T}$ is a σ-algebra on Y and that $(Y, \mathcal{T}, \nu)$ is a measure space.

Problem 2: Let $(X, \mathcal{S}, \mu)$ be a measure space. Let

$$\mathcal{S}_\mu = \{E \in \mathcal{P}(X) : \text{There exists } A, B \in \mathcal{S} \text{ with } A \subseteq E \subseteq B \text{ and } \mu(B \backslash A) = 0\}.$$

Show that $\mathcal{S}_\mu$ is a σ-algebra containing $\mathcal{S}$.
Aside: The idea here is that if $(X, \mathcal{S}, \mu)$ is *not* complete, and we want to expand the σ-algebra to make it complete, then we would have to throw in all sets E with the above property.

Problem 3: Let μ^* be an outer measure on a nonempty set X. Show that if $E \subseteq X$ is such that $\mu^*(E) = 0$, then

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A)$$

for all $A \subseteq X$. In other words, every set with outer measure 0 satisfies the Carathéodory condition for measurability.