Homework 7: Due Friday, October 27

Problem 1: Determine, with explanation, the value of

$$\lim_{n\to\infty}\int_0^1 \left(\frac{nx}{1+n} + 3\sin(x^2 n)\cdot x^n\right).$$

Problem 2: Let $f\colon [a,b]\to\mathbb{R}$ be a nonnegative measurable function. Let $c,d\in\mathbb{R}$ with both $c>0$ and $d>0$. Show that if $\int_a^b f < d$, then $m(\{x\in[a,b] : f(x)\geq c\}) < \frac{d}{c}$.

Problem 3: Let $f\colon [a,b]\to\mathbb{R}$ be a measurable function.
a. Show that for all $\varepsilon>0$, there exists $K\in\mathbb{R}$ such that $m(\{x\in[a,b] : f(x)\notin[-K,K]\})<\varepsilon$.
b. Show that for all $\varepsilon>0$ and all $K\in\mathbb{R}$, there exists a simple function $\varphi\colon[a,b]\to\mathbb{R}$ such that $|f(x)-\varphi(x)|<\varepsilon$ for all $x\in[a,b]$ with $-K\leq f(x)\leq K$.
c. Show that for all $\varepsilon>0$, there exists a simple function $\varphi\colon[a,b]\to\mathbb{R}$ such that

$$m(\{x\in[a,b] : |f(x)-\varphi(x)|\geq\varepsilon\})<\varepsilon.$$

Problem 4: Let $f\colon\mathbb{R}\to\mathbb{R}$ be a nonnegative integrable function, i.e. assume that f is nonnegative, measurable, and that $\int f<\infty$. Define $F\colon\mathbb{R}\to\mathbb{R}$ by letting

$$F(x)=\int_{(-\infty,x]} f.$$

Show that F is continuous.

Problem 5:
a. Let $f_1, f_2, \dots$ be a sequence of nonnegative measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\sum_{k=1}^{\infty} f_k(x)$ converges for all $x\in\mathbb{R}$. Let $g(x)=\sum_{k=1}^{\infty} f_k(x)$. Show that

$$\int g = \sum_{k=1}^{\infty}\left(\int f_k\right).$$

b. Let $g\colon\mathbb{R}\to\mathbb{R}$ be a nonnegative measurable function, and let $A_1, A_2, \dots$ be a sequence of pairwise disjoint measurable subsets of $\mathbb{R}$. Show that

$$\int_{\bigcup\limits_{k=1}^{\infty} A_k} g = \sum_{k=1}^{\infty}\left(\int_{A_k} g\right).$$

Problem 6: Suppose that $f\colon\mathbb{R}\to\mathbb{R}$ is integrable, i.e. f is measurable and both $\int f^+<\infty$ and $\int f^-<\infty$. Let $g\colon\mathbb{R}\to\mathbb{R}$ be a bounded measurable function. Show that $f\cdot g\colon\mathbb{R}\to\mathbb{R}$ is integrable.