Homework 6: Due Friday, October 6

Problem 1: Show that if $N \subseteq \mathbb{R}$ and $m^*(N) = 0$, then in $\mathbb{R}^2$ we have $m^*(\mathbb{R} \times N) = 0$.

Problem 2: Let $A_1, A_2, A_3, \ldots$ be a sequence of measurable sets in $\mathbb{R}^n$, and let

$$B = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k.$$

a. Show that $B = \{x \in \mathbb{R}^n : \text{There are infinitely many } k \in \mathbb{N}^+ \text{ with } x \in A_k\}$.
b. Show that B is measurable.
c. Show that if $\sum_{k=1}^{\infty} m(A_k) < \infty$, then $m(B) = 0$.
Aside: Part (c) is known as the Borel-Cantelli Lemma.

Problem 3: Let $k \in \mathbb{R}$ with $k > 0$. Show that any collection of pairwise disjoint measurable subsets of $[-k, k]$, each of which has positive measure, must be countable.
Note: It is possible to extend from this bounded case to the general case of having pairwise disjoint measurable subsets of $\mathbb{R}$, or even $\mathbb{R}^n$, but you do not have to do that here.

Problem 4: Let $\mathcal{P}(\mathbb{R})$ be the power set of $\mathbb{R}$, i.e. the set of all subsets of $\mathbb{R}$. Let $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R})$ be the σ-algebra generated by the open sets and the sets of measure zero, i.e. $\mathcal{S}$ is the smallest σ-algebra that contains every open set and every set of measure 0. Show that $\mathcal{S}$ is the collection of measurable sets.

Problem 5: Let $I = [c, d]$ for some $c, d \in \mathbb{R}$ with $c < d$.
a. Show that if $f \colon I \to \mathbb{R}$ is (weakly) increasing, i.e. if $f(x) \leq f(y)$ whenever $x < y$, then f is measurable.
b. Let $f, g \colon I \to \mathbb{R}$ be measurable functions, and assume that $g(x) \neq 0$ for all $x \in I$. Show that $\frac{f}{g}$ is a measurable function.

Problem 6: Give an example (with proof) of a function $f \colon [0, 1] \to \mathbb{R}$ such that f is not measurable, but $\{x \in \mathbb{R} : f(x) = a\}$ is measurable for each $a \in \mathbb{R}$.