Homework 5: Due Friday, September 29

Problem 1: Suppose that $A, N \subseteq \mathbb{R}$ with $m^*(A) < \infty$ and $m^*(N) = 0$. Show that in $\mathbb{R}^2$, we have $m^*(A \times N) = 0$.

Problem 2: Show that if $A, B \subseteq \mathbb{R}^n$ are measurable, then

$$m(A \cup B) + m(A \cap B) = m(A) + m(B).$$

Keep in mind that some of these values might be infinite, so be sure that your argument handles that situation in some way.

Problem 3: Recall that given two sets A and B, the *symmetric difference* of A and B is the set

$$A \triangle B = \{x : x \text{ is an element of exactly one of } A \text{ or } B\}.$$

Let $A \subseteq \mathbb{R}^n$. Show that if there is a measurable set $B \subseteq \mathbb{R}^n$ with $m^*(A \triangle B) = 0$, then A is measurable.

Problem 4: Let $A \subseteq \mathbb{R}^n$ with $m^*(A) < \infty$. Define

$$c = \sup\{m(F) : F \text{ is a closed set with } F \subseteq A\}$$

and

$$d = \inf\{m(G) : G \text{ is an open set with } A \subseteq G\}.$$

a. Show that $c \leq d$.
b. Show that A is measurable if and only if $c = d$, and that in this case the common value is $m(A)$.

Problem 5: A sequence $A_1, A_2, A_3, \ldots$ of measurable sets in $\mathbb{R}^n$ is *almost disjoint* if $m(A_i \cap A_j) = 0$ whenever $i \neq j$. Show that if $A_1, A_2, A_3, \ldots$ is an almost disjoint sequence of measurable sets in $\mathbb{R}^n$, then $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$.

Problem 6: Given $A, B \subseteq \mathbb{R}$, let $A + B = \{a + b : a \in A, b \in B\}$.
a. Show that if A is open, then $A + B$ is open.
b. Show that if A is closed and B is compact, then $A + B$ is closed.
c. Show that if A and B are both closed, then $A + B \in \mathcal{F}_\sigma$, i.e. $A + B$ can be written as a countable union of closed sets.
Aside: It is an interesting exercise to construct an example of closed sets $A, B \subseteq \mathbb{R}$ where $A+B$ is not closed.