Homework 4: Due Friday, September 22

Problem 1: As mentioned in class, *(Jordan) outer content* c^* is defined in the same way as m^*, but only using finite covers. In other words

$$c^*(A) = \inf\left\{\sum_{k=1}^{\ell} v(I_k) : A \subseteq \bigcup_{k=1}^{\ell} I_k \text{ and each } I_k \text{ is a closed interval}\right\}.$$

Notice that $m^*(A) \leq c^*(A)$ for every set $A \subseteq \mathbb{R}^n$.
a. Show that $c^*(A) = c^*(\mathsf{cl}(A))$ for every set $A \subseteq \mathbb{R}^n$.
b. Give an example of a set $A \subseteq \mathbb{R}$ with $m^*(A) \neq m^*(\mathsf{cl}(A))$.

Problem 2: Let $c, d \in \mathbb{R}$ with $c < d$, and let $N \subseteq (c, d)$ with $m^*(N) = 0$. Let $A = (c, d)\backslash N$.
a. Show that A is uncountable.
b. Show that A is dense in (c, d), i.e. for all $x, y \in \mathbb{R}$ with $c < x < y < d$, we have $A \cap (x, y) \neq \emptyset$.

Problem 3: Let A be the subset of $[0, 1]$ consisting of those numbers that do not contain the digit 3 in their decimal representations. Show that $m^*(A) = 0$.

Problem 4: Given $A \subseteq \mathbb{R}$, we define $A^2 = \{a^2 : a \in A\}$. Show that if $A \subseteq \mathbb{R}$ and $m^*(A) = 0$, then $m^*(A^2) = 0$.
Hint: It is easier to do this if you have some bounds on the elements of A. First assume that $A \subseteq [0, n]$ for some $n \in \mathbb{N}^+$, and then work with a general set.

Problem 5: Show that if $A \subseteq \mathbb{R}$ is measurable, then for any $t \in \mathbb{R}$, the set $tA = \{ta : a \in A\}$ is measurable.

Problem 6: Given two sets A and B, let $A + B = \{a + b : a \in A, b \in B\}$. Show that $C + C = [0, 2]$, where C is the Cantor Set.
Hint: Using the notation from Definition 1.21 from the Metric Space notes, first show that $C_n + C_n = [0, 2]$ for all $n \in \mathbb{N}$. It may help to notice that $C_{n+1} = \frac{1}{3}C_n \cup (\frac{2}{3} + \frac{1}{3}C_n)$ for all $n \in \mathbb{N}$ (you do not need to prove this, but do think about why it is true).
Aside: Recall that $m^*(C) = 0$, so it is pretty surprising that we can obtain a set with positive outer measure by adding together two sets with outer measure 0.