Homework 3: Due Friday, September 15

Problem 1: Let X be a set, and let $d_1, d_2 \colon X^2 \to \mathbb{R}$ be two (potentially different) metrics on X. Given $x \in X$ and $\varepsilon > 0$, let $B_{1,\varepsilon}(x) = \{y \in X : d_1(x,y) < \varepsilon\}$ and let $B_{2,\varepsilon}(x) = \{y \in X : d_2(x,y) < \varepsilon\}$.
a. Suppose that for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $B_{1,\delta}(x) \subseteq B_{2,\varepsilon}(x)$. Show that if $D \subseteq X$ is open in (X, d_2), then D is open in (X, d_1).
b. Let $X = \mathbb{R}^2$, let

$$d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

and let

$$d_2((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Show that (X, d_1) and (X, d_2) have the same open sets, i.e. that for all $D \subseteq X$, we have that D is open in (X, d_1) if and only if D is open in (X, d_2).

Problem 2: Let (X, d) be a metric space. Show that every closed subset of X can be written as a countable intersection of open sets.
Hint: Given a closed set A, consider the sets $\{x \in X : \text{There exists } a \in A \text{ with } d(a,x) < \varepsilon\}$ for various ε.

Problem 3: Let (X, d_1) and (Y, d_2) be metric spaces, and let $f \colon X \to Y$. Show that f is continuous if and only if $f^{-1}(A)$ is closed in X whenever A is closed in Y.

Problem 4: Suppose that $f \colon [0,1] \to \mathbb{R}$ is continuous, that $f(x) \geq 0$ for all $x \in [0,1]$, and that there exists $x \in [0,1]$ with $f(x) \neq 0$. Show that

$$\int_0^1 f(x)\ dx > 0.$$

Note: This result is needed to prove that each of the examples with $X = C[0,1]$ on the first page of the metric space notes satisfy Property 1.

Problem 5: Show that if $A, N \subseteq \mathbb{R}^n$ and $m^*(N) = 0$, then $m^*(A \cup N) = m^*(A)$.

Problem 6: Let $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$.
a. Let $A + t = \{a + t : a \in A\}$ be the translation of A by t. Show that $m^*(A + t) = m^*(A)$.
b. Let $tA = \{ta : a \in A\}$ be the scaling of A by t. Show that $m^*(tA) = |t| \cdot m^*(A)$.
Note: The argument for (a) generalizes in a straightforward way when $A \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}^n$. For (b), if $A \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}$, the result is $m^*(tA) = |t|^n \cdot m^*(A)$ (think about why).