Homework 2: Due Friday, September 8

Problem 1: Either prove or give a counterexample to each of the following:
a. If (X, d) is a metric space and $A, B \subseteq X$, then $\mathsf{int}(A \cup B) = \mathsf{int}(A) \cup \mathsf{int}(B)$.
b. If (X, d) is a metric space and $A, B \subseteq X$, then $\mathsf{int}(A \cap B) = \mathsf{int}(A) \cap \mathsf{int}(B)$.

Problem 2: Let (X, d) be a metric space, let $x \in X$, and let $r \in \mathbb{R}$.
a. Show that $\{y \in X : d(x, y) \leq r\}$ is a closed set.
b. Show that $\{y \in X : d(x, y) = r\}$ is a closed set.

Problem 3: Let $A \subseteq \mathbb{R}$ be a bounded open set, and let $x \in A$. Let $c = \inf\{b \in \mathbb{R} : b < x \text{ and } (b, x) \subseteq A\}$ and let $d = \sup\{b \in \mathbb{R} : b > x \text{ and } (x, b) \subseteq A\}$. Show that $c \notin A$, that $d \notin A$, and that $(c, d) \subseteq A$.

Problem 4: Prove that a nonempty compact subset of $\mathbb{R}$ must have a maximum and a minimum.

Problem 5: Let (X, d) be the metric space where $X = \mathbb{N}^+$ and $d \colon X \times X \to \mathbb{R}$ is given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Give an example, with proof, of a closed and bounded subset of X that is *not* compact.

Problem 6: In class, we defined a metric space whose elements were infinite sequences of 0s and 1s. More formally, we let X be the set of all functions $f \colon \mathbb{N} \to \{0, 1\}$, and defined $d \colon X \times X \to \mathbb{R}$ as follows. Given $f, g \colon \mathbb{N} \to \{0, 1\}$, let $d(f, g) = 0$ if $f(n) = g(n)$ for all $n \in \mathbb{N}$, and otherwise let $d(f, g) = 2^{-m}$, where $m = \min\{n \in \mathbb{N} : f(n) \neq g(n)\}$.
a. Show that $d(f, h) \leq \max\{d(f, g), d(g, h)\}$ for all $f, g, h \in X$.
b. Show that $B_\varepsilon(f)$ is closed for every $f \in X$ and every $\varepsilon > 0$ (we know from Proposition 1.17 that each of these sets is also open).
Note: The triangle inequality condition on metric spaces follows immediately from part (a). Metric spaces with this stronger property are called *ultrametric spaces*. The metric space in Problem 5 is also an ultrametric space.