Homework 1: Due Friday, September 1

Problem 1: Let $a_n \in \mathbb{R}$ with $a_n \geq 0$ for all $n \in \mathbb{N}^+$. Let

$$B = \left\{ \sum_{n=1}^{N} a_n : N \in \mathbb{N}^+ \right\} \qquad \text{and} \qquad C = \left\{ \sum_{n \in F} a_n : F \in \mathcal{P}_{\mathsf{fin}}(\mathbb{N}^+) \right\},$$

where $\mathcal{P}_{\mathsf{fin}}(\mathbb{N}^+)$ is the set of all *finite* subsets of $\mathbb{N}^+$.

a. Show that $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if B is bounded above, and that in this case $\sum\limits_{n=1}^{\infty} a_n = \sup B$.

b. Show that B is bounded above if and only if C is bounded above, and that in this case $\sup B = \sup C$.

Note: It follows that $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if C is bounded above, and that in this case $\sum\limits_{n=1}^{\infty} a_n = \sup C$.

Problem 2: A *normed vector space* is a vector space V together with a function $|| \cdot || \colon V \to \mathbb{R}$ having the following properties:

1. $||\vec{0}|| = 0$ and $||\vec{v}|| > 0$ for all $\vec{v} \in V \backslash \{\vec{0}\}$.
2. $||c \cdot \vec{v}|| = |c| \cdot ||\vec{v}||$ for all $c \in \mathbb{R}$ and all $\vec{v} \in V$.
3. $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$ for all $\vec{v}, \vec{w} \in V$.

Suppose that V is a normed vector space, and define $d \colon V^2 \to \mathbb{R}$ by letting $d(\vec{v}, \vec{w}) = ||\vec{v} - \vec{w}||$. Show that (V, d) is a metric space.

Problem 3: Show that the vector space $\mathbb{R}^n$ with $||(x_1, x_2, \dots, x_n)|| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ is a normed vector space.

Note: Using Problem 2, it follows that Example 3 in the course notes is really a metric space.

Problem 4: Suppose that $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences from $\mathbb{R}$ under the usual metric. Assume that $\lim\limits_{n \to \infty} x_n = a$ and $\lim\limits_{n \to \infty} y_n = b$. Show that in $\mathbb{R}^2$ with the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

we have $\lim\limits_{n \to \infty} (x_n, y_n) = (a, b)$.

Note: The corresponding result is true in $\mathbb{R}^n$ (your argument should naturally generalize), but the notation becomes a bit cumbersome there.

Problem 5: Let (X, d) be a metric space, and let $A \subseteq X$ be nonempty. Show that the following are equivalent:

1. There exists $a \in A$ and $r \in \mathbb{R}$ with $A \subseteq B_r(a)$.
2. There exists $r \in \mathbb{R}$ such that for all $a, b \in A$, we have $d(a, b) < r$.
3. For all $a \in A$, there exists $r \in \mathbb{R}$ with $A \subseteq B_r(a)$.

If A satisfies any (and hence all) of the above, then A is called *bounded* (the empty set is also considered bounded).

Problem 6: Consider the metric space $\mathbb{R}^n$ with

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

Show that for all $\vec{v} \in \mathbb{R}^n$ and all $\varepsilon > 0$, there exists $\vec{q} \in \mathbb{Q}^n$ with $d(\vec{v}, \vec{q}) < \varepsilon$.
Hint: Make repeated use of the Density of $\mathbb{Q}$ in $\mathbb{R}$.