Homework 9: Due Friday, November 9

Problem 1: Using the definition of a limit (i.e. do not use any of our powerful results), show that $\lim_{x \to 1/2} \frac{1}{x} = 2$.

Problem 2: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, and let $a, c, \ell \in \mathbb{R}$. Suppose that $\lim_{x \to c} f(x) = \ell$.
a. Define $g\colon \mathbb{R} \to \mathbb{R}$ by letting $g(x) = f(x + a)$. Show that $\lim_{x \to c-a} g(x) = \ell$.
b. Define $g\colon \mathbb{R} \to \mathbb{R}$ by letting $g(x) = f(ax)$. Show that if $a \neq 0$, then $\lim_{x \to c/a} g(x) = \ell$.

Problem 3: Let $f\colon A \to \mathbb{R}$ be a function and let $c, \ell \in \mathbb{R}$. Assume that for all $\delta > 0$, the sets $(c, c + \delta) \cap A$ and $(c - \delta, c) \cap A$ are both nonempty. We define left-sided and right-sided limits as follows:

1. We define $\lim_{x \to c^-} f(x) = \ell$ to mean that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, we have $|f(x) - \ell| < \varepsilon$.
2. We define $\lim_{x \to c^+} f(x) = \ell$ to mean that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, we have $|f(x) - \ell| < \varepsilon$.

Show that $\lim_{x \to c} f(x) = \ell$ if and only if both $\lim_{x \to c^-} f(x) = \ell$ and $\lim_{x \to c^+} f(x) = \ell$.

Definition: Let $f\colon A \to \mathbb{R}$ be a function and let $c, \ell \in \mathbb{R}$.

1. Suppose that c is a limit point of A. We define $\lim_{x \to c} f(x) = \infty$ to mean that for every $z \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, we have $f(x) > z$.
2. Suppose that for all $y \in \mathbb{R}$, there exists $x \in A$ with $x > y$. We define $\lim_{x \to \infty} f(x) = \ell$ to mean that for every $\varepsilon > 0$, there exists $z \in \mathbb{R}$ such that for all $x \in A$ with $x > z$, we have $|f(x) - \ell| < \varepsilon$.

Problem 4: Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Problem 5: Let $f, g\colon \mathbb{R} \to \mathbb{R}$ and let $\ell, m \in \mathbb{R}$. Suppose that $\lim_{x \to \infty} f(x) = \ell$ and $\lim_{x \to \infty} g(x) = m$. Show that $\lim_{x \to \infty} (f + g)(x) = \ell + m$.

Problem 6: Let

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is continuous at 0, but not not continuous at any $c \neq 0$.