Homework 8: Due Friday, November 2

Problem 1: Let $A, B \subseteq \mathbb{R}$ be open sets. Show that the set $A + B = \{a + b : a \in A, b \in B\}$ is open.

Problem 2: Show that each of the following sets is *not* compact by explicitly giving an open cover that does *not* have a finite subcover. Briefly explain in each case (no need for a full detailed proof).
a. $\mathbb{Q} \cap [0, 2]$.
b. The set

$$\left\{ \sum_{k=0}^{n} \frac{1}{2^k} : n \in \mathbb{N} \right\} = \left\{ 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, \dots \right\}.$$

Problem 3: Let A be a nonempty compact set. Show that A has a maximum element, i.e. that there exists $c \in A$ such that $a \leq c$ for all $a \in A$.
Aside: A similar argument shows that every nonempty compact set has a minimum. This problem illustrates one of the many ways that compact sets behave a lot like finite sets. After all, every nonempty finite set also has a maximum and a minimum.

Problem 4: Suppose that we have a nonempty compact set K_n for each $n \in \mathbb{N}^+$ and that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots.$$

We know from Problem 3 that each K_n has a maximum, so we can let $a_n = \max(K_n)$ for each $n \in \mathbb{N}^+$.
a. Show that $\langle a_n \rangle$ is decreasing and bounded below.
b. By the Monotone Convergence Theorem, we know that $\langle a_n \rangle$ converges. Let $b = \lim_{n\to\infty} a_n$. Show that $b \in K_n$ for all $n \in \mathbb{N}^+$. In particular, the intersection $\bigcap_{n\in\mathbb{N}^+} K_n$ is nonempty.

Problem 5: Give an example of a collection of nonempty closed sets $\{A_n : n \in \mathbb{N}^+\}$ such that

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$$

and where $\bigcap_{n\in\mathbb{N}^+} A_n = \emptyset$.
Note: Thus, the conclusion of Problem 4b does not hold if we replace the *compact* sets with *closed* sets.