Homework 6: Due Friday, October 12

Problem 1: Determine whether the following series converge or diverge. Justify your answers!

$$\text{a. } \sum_{n=1}^{\infty} \frac{(n+1)^2}{4n(n+2)} \qquad \text{b. } \sum_{n=1}^{\infty} \frac{n^2}{7^n} \qquad \text{c. } \sum_{n=1}^{\infty} \frac{2^n+n}{3^n-2}$$

Problem 2: In class, and on p. 59-60 of the course notes, we talked about trying to approximate $n!$. In this problem, we establish some of the relationships betwen $n!$ and $(\frac{n}{a})^n$ that we alluded to in that discussion. Although we have not yet established this fact, you might know from Calculus that $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$. Feel free to use that information in this problem.

a. Suppose that $a \in \mathbb{R}$ and $0 < a < e$. Show that $\sum_{n=1}^{\infty} \frac{n!}{(n/a)^n}$ converges. Conclude that $\lim\limits_{n\to\infty} \frac{n!}{(n/a)^n} = 0$.

b. Suppose that $a \in \mathbb{R}$ and $a > e$. Show that $\sum_{n=1}^{\infty} \frac{(n/a)^n}{n!}$ converges. Conclude that $\lim\limits_{n\to\infty} \frac{(n/a)^n}{n!} = 0$.

Problem 3: In our definition of a series, we looked at the partial sums s_n formed by adding the firm n terms of the series. But we can consider other approaches. Let $\langle a_n \rangle$ be a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}^+$. Let

$$B = \left\{ \sum_{k=1}^{n} a_k : n \in \mathbb{N}^+ \right\} \qquad \text{and} \qquad C = \left\{ \sum_{k \in F} a_k : F \in \mathcal{P}_{fin}(\mathbb{N}^+) \right\},$$

where $\mathcal{P}_{fin}(\mathbb{N}^+)$ is the set of all finite subsets of $\mathbb{N}^+$. In other words, $B = \{s_n : n \in \mathbb{N}^+\}$ is the set of the usual partial sums of $\langle a_n \rangle$, and C is the set of *all* finite sums of the terms of the sequence, whether they consist of the first n terms in order, or any other collection of finitely many terms. Show that B is bounded above if and only if C is bounded above, and that in this case $\sup B = \sup C$.

Note: We know from Proposition 3.2.1 that $\sum_{n=1}^{\infty} a_n$ converges if and only if B is bounded above, and that in this case we have $\sum_{n=1}^{\infty} a_n = \sup B$. It follows that $\sum_{n=1}^{\infty} a_n$ converges if and only if C is bounded above, and that in this case $\sum_{n=1}^{\infty} a_n = \sup C$.

Problem 4:

a. Give an example of a sequence $\langle a_n \rangle$ such that $\lim\limits_{n\to\infty} |a_{n+1} - a_n| = 0$ yet $\langle a_n \rangle$ diverges.

b. Suppose that $\sum_{n=1}^{\infty} b_n$ converges and $\langle a_n \rangle$ is a sequence such that $|a_{n+1} - a_n| < b_n$ for every $n \in \mathbb{N}^+$. Show that $\langle a_n \rangle$ converges.

Definition: Let $\langle a_n \rangle$ be a sequence such that $a_n \neq 0$ for every $n \in \mathbb{N}^+$. Define a new sequence $\langle p_n \rangle$ by letting $p_n = a_1 a_2 \cdots a_n$ for every $n \in \mathbb{N}^+$. We call $\langle p_n \rangle$ the sequence of *partial products* of $\langle a_n \rangle$. We say that the *infinite product* $\prod_{n=1}^{\infty} a_n$ *converges* if the sequence $\langle p_n \rangle$ converges and $\lim_{n\to\infty} p_n \neq 0$ (we forbid a limit of 0 for convenience because 0 behaves very badly with respect to multiplication). In this case we also write $\prod_{n=1}^{\infty} a_n$ to denote the number $\lim\limits_{n\to\infty} p_n$.

Problem 5:

a. Define a sequence $\langle a_n \rangle$ by letting

$$a_n = 1 - \frac{1}{(n+1)^2}$$

for every $n \in \mathbb{N}^+$. Find a simple expression for the partial product p_n, and use it to show that $\prod_{n=1}^{\infty} a_n$ converges and to find the corresponding value.

b. Let $\langle a_n \rangle$ be a sequence with $a_n \neq 0$ for all $n \in \mathbb{N}^+$. Show that if $\prod_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 1$.