Homework 5: Due Friday, October 5

Problem 1: Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences. Suppose that $\langle a_n \rangle$ diverges to ∞ and that $\langle b_n \rangle$ is bounded below. Show that $\langle a_n + b_n \rangle$ diverges to ∞.

Problem 2: Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be Cauchy sequences. Using only the definition (i.e. do not use the fact that a sequence converges if and only if it is Cauchy), show that $\langle a_n + b_n \rangle$ is a Cauchy sequence.

Problem 3: Let $\langle a_n \rangle$ be a bounded sequence. Define two new sequence $\langle b_n \rangle$ and $\langle c_n \rangle$ as follows:

- For each $n \in \mathbb{N}^+$, let $b_n = \sup\{a_k : k \geq n\}$.
- For each $n \in \mathbb{N}^+$, let $c_n = \inf\{a_k : k \geq n\}$.

a. Show that $\langle b_n \rangle$ is decreasing and bounded below.
b. Show that $\langle c_n \rangle$ is increasing and bounded above.
c. Using the Monotone Convergence Theorem, it follows that both $\langle b_n \rangle$ and $\langle c_n \rangle$ converge. Show that $\lim\limits_{n\to\infty} c_n \leq \lim\limits_{n\to\infty} b_n$.
Aside: It can be shown that $\lim\limits_{n\to\infty} b_n$ is the largest cluster point of $\langle a_n \rangle$ and that $\lim\limits_{n\to\infty} c_n$ is the smallest cluster point of $\langle a_n \rangle$.

Interlude: Suppose that $x, y \in \mathbb{R}$ with $x, y \geq 0$. The normal average of x and y is $\frac{x+y}{2}$, and this quantity is sometimes known as the *arithmetic mean* of x and y. Another way to "average" the values x and y is to use the number $\sqrt{xy}$. Geometrically, if we form a rectangle with side lengths x and y, then $\sqrt{xy}$ is the side length of the square that has the same area. The quantity $\sqrt{xy}$ is called the *geometric* mean of x and y.

A fundamental result known as the AM-GM inequality says that for all $x, y \in \mathbb{R}$ with $x, y \geq 0$, we have $\sqrt{xy} \leq \frac{x+y}{2}$, with equality if and only if $x = y$. In other words, the geometric mean is always less then or equal to the arithmetic mean. There are many proofs of this fact, but perhaps the most straightforward is to notice that $(x-y)^2 \geq 0$, and then expand the left hand side and rearrange to conclude that $(x+y)^2 \geq 4xy$. One then conclude that $x + y \geq 2\sqrt{xy}$. Feel free to take the result as known, or to work out the details.

Problem 4: Let $a_1, b_1 \in \mathbb{R}$ with $0 \leq a_1 < b_1$. Define two sequence $\langle a_n \rangle$ and $\langle b_n \rangle$ recursively by starting with a_1 and b_1, and then recursively defining

$$a_{n+1} = \sqrt{a_n b_n} \qquad\qquad b_{n+1} = \frac{a_n + b_n}{2}$$

for all $n \in \mathbb{N}^+$. In other words, a_2 is the geometric mean of a_1 and b_1, while b_2 is the arithmetic mean of a_1 and b_1. We then continue on to let a_3 (resp. b_3) be the geometric (resp. arithmetic) mean of a_2 and b_2, etc.
a. Show that $\langle a_n \rangle$ and $\langle b_n \rangle$ both converge.
b. Show that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.