Homework 4: Due Friday, September 28

Problem 1: Determine whether each of the following sequences converges or diverges. In the case of convergence, find the limit (with proof).
a. $\langle \sqrt{n^2+2n+1} - n \rangle$.
b. $\langle \sqrt{n^2+1} - n \rangle$.
c. $\langle \frac{4n^2+5n-1}{3n^2-2n+7} \rangle$.

Problem 2:
a. Give an example of a sequence $\langle a_n \rangle$ such that $\langle |a_n| \rangle$ converges, but $\langle a_n \rangle$ does not.
b. Show that if $\langle a_n \rangle$ converges, then $\langle |a_n| \rangle$ converges.

Problem 3: Define a sequence recursively as follows. Let $a_1 = \sqrt{2}$ and let $a_{n+1} = \sqrt{2a_n}$ for each $n \in \mathbb{N}^+$. Thus, our sequence looks like

$$\sqrt{2} \quad \sqrt{2\sqrt{2}} \quad \sqrt{2\sqrt{2\sqrt{2}}} \quad \cdots$$

a. Show that $\langle a_n \rangle$ is increasing and bounded above by 2.
b. Using the Monotone Convergence Theorem, we know that $\langle a_n \rangle$ converges. Calculate the limit of $\langle a_n \rangle$.

Problem 4:
a. Suppose that $\langle a_n \rangle$ converges to ℓ. Define a new sequence $\langle b_n \rangle$ by letting

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

for all $n \in \mathbb{N}^+$. In other words, b_n is the result of averaging the first n terms of $\langle a_n \rangle$. Show that $\langle b_n \rangle$ converges to ℓ.
b. Give an example of a sequence $\langle a_n \rangle$ such that $\langle a_n \rangle$ diverges, but the sequence $\langle b_n \rangle$ defined in part (a) converges.

Definition: Let $\langle a_n \rangle$ be a sequence. We say that ℓ is a *cluster point* of $\langle a_n \rangle$ if for every $\varepsilon > 0$ and every $N \in \mathbb{N}$, there exists $n \geq N$ with $|a_n - \ell| < \varepsilon$.

Notice that we have simply flipped the quantifiers on the N and the n from the definition of a limit. Intuitively, ℓ is a cluster point of $\langle a_n \rangle$ if no matter how small an error we are given, and far out we are restricted to look in the sequence, we can always find at least one element of the sequence that is sufficiently close to ℓ. Alternatively, no matter how an error we are given, we can always find infinitely many elements of the sequences sufficiently close to ℓ. For example, notice that both 1 and -1 are cluster points of $\langle (-1)^n \rangle$.

Problem 5:
a. Give a counterexample to the following statement: If ℓ is a cluster point of $\langle a_n \rangle$ and m is a cluster point of $\langle b_n \rangle$, then $\ell + m$ is a cluster point of $\langle a_n + b_n \rangle$.
b. Show that if $\langle a_n \rangle$ converges to ℓ, and m is a cluster point of $\langle b_n \rangle$, then $\ell + m$ is a cluster point of $\langle a_n + b_n \rangle$.

Problem 6:
a. Give an example of a sequence for which every $n \in \mathbb{N}$ is a cluster point.
b. Give an example of a sequence for which every $q \in \mathbb{Q}$ is a cluster point.
c. Let $\langle a_n \rangle$ be a sequence for which every $q \in \mathbb{Q}$ is a cluster point. Show that every $x \in \mathbb{R}$ is a cluster point of $\langle a_n \rangle$. In light of part (b), it follows that there exists a sequence for which every real number is a cluster point.