Homework 3: Due Friday, September 21

Problem 1: Without using our work on countable and uncountable sets, show that if $c, d \in \mathbb{R}$ and $c < d$, then there exists $z \in \mathbb{R}\backslash\mathbb{Q}$ with $c < z < d$.
Hint: Pick your favorite irrational number, and think about how to use it in conjunction with the Density of $\mathbb{Q}$ in $\mathbb{R}$.

Problem 2: Show that if A and B are countable, then $A \times B = \{(a, b) : a \in A, b \in B\}$ is countable.

Problem 3: Show that if $c, d \in \mathbb{R}$ with $c < d$, then $(c, d) \cap (\mathbb{R}\backslash\mathbb{Q})$ is uncountable, i.e. every nontrivial open interval contains uncountably many irrationals.

Problem 4: Let $A \subseteq \mathbb{R}$ be uncountable. Show that there is a bounded set $B \subseteq A$ such that B is uncountable.

Problem 5: We defined a set A to be countably infinite if there exists a bijection $f : \mathbb{N} \to A$. More generally, we say that two sets A and B have the same cardinality if there exists a bijection $f : A \to B$.
a. Let $a \in \mathbb{R}$. Show that $(0, \infty) = \{x \in \mathbb{R} : x > 0\}$ and $(a, \infty) = \{x \in \mathbb{R} : x > a\}$ have the same cardinality by finding (with proof) an explicit bijection $f : (0, \infty) \to (a, \infty)$.
b. Let $a, b \in \mathbb{R}$ with $a < b$. Show that the open intervals $(0, 1)$ and (a, b) have the same cardinality by finding (with proof) an explicit bijection $f : (0, 1) \to (a, b)$.
c. Show that $(0, 1)$ and $(1, \infty) = \{x \in \mathbb{R} : x > 1\}$ have the same cardinality by finding (with proof) an explicit bijection $f : (0, 1) \to (1, \infty)$.
Note: Since the composition of bijections is a bijection, and the inverse of a bijection is a bijection, it follows that (a, b) and (c, ∞) have the same cardinality whenever $a, b, c \in \mathbb{R}$ and $a < b$. It also possible to show that all nontrivial closed intervals have the same cardinality as all nontrivial open intervals, but constructing an explicit bijection is much harder.

Problem 6: A number $b \in \mathbb{R}$ is called an *almost upper bound* of a set $A \subseteq \mathbb{R}$ if $\{a \in A : a > b\}$ is finite. Notice that an upper bound of A is just a number such that this set is empty.
a. Give an example of an infinite set $A \subseteq \mathbb{R}$ and a number $b \in \mathbb{R}$ such that b is an almost upper bound of A, but not an upper bound of A.
b. Let $A \subseteq \mathbb{R}$ be a bounded infinite set. Let $B = \{b \in \mathbb{R} : b \text{ is an almost upper bound of } A\}$. Show that B is nonempty and bounded below.
c. Give an example of a bounded infinite set $A \subseteq \mathbb{R}$ such that $\{a \in A : a > \inf B\}$ is infinite (where B is defined as in part (b)). In particular, $\inf B$ might not be an almost upper bound of A.
d. Let $A \subseteq \mathbb{R}$ be a bounded infinite set. Show that $\{a \in A : a > \inf B\}$ is countable (where again B is defined as in part (b)).