Homework 2: Due Friday, September 14

Problem 1: Show that if $A \subseteq \mathbb{R}$ is a bounded nonempty set, then $\inf A \leq \sup A$.

Problem 2: Let $B \subseteq \mathbb{R}$ be a nonempty set that is bounded below. Let

$$A = \{a \in \mathbb{R} : a \text{ is a lower bound of } B\}.$$

a. Show that A is nonempty and bounded above.
b. Show that $\sup A$ is a greatest lower bound of B.
Note: This problem gives another proof of the fact that every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound, so avoid using the existence of greatest lower bounds in your argument.

Problem 3: Let $A, B \subseteq \mathbb{R}$ be nonempty sets.
a. Assume that $a \leq b$ whenever $a \in A$ and $b \in B$. Show that A is bounded above, that B is bounded below, and that $\sup A \leq \inf B$.
b. By giving an explicit counterexample, show that if we instead assume that $a < b$ whenever $a \in A$ and $b \in B$, then we can *not* necessarily conclude that $\sup A < \inf B$.
Aside: When we defined $A = \{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$ and $B = \{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$ near the beginning of class, we showed that $a \leq b$ whenever $a \in A$ and $b \in B$. At the time, we talked about how we "should" have a number that serves as a dividing line between the two sets. Part (a) of this problem says that reals always contain such dividing lines, because we can always take $\sup A$ (or $\inf B$) as a value that works.

Problem 4: Let $A, B \subseteq \mathbb{R}$ be nonempty sets that are bounded above. Show that $A \cup B$ is bounded above and that $\sup(A \cup B) = \max(\sup A, \sup B)$.

Problem 5: Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above and let $c \in \mathbb{R}$ with $c > 0$. Let $B = \{c \cdot a : a \in A\}$. Show that B is bounded above and that $\sup(B) = c \cdot \sup(A)$.

Problem 6: We know that $\mathbb{Q} \subseteq \mathbb{R}$. We define the set of irrationals to be $\mathbb{R}\backslash\mathbb{Q} = \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$. Now we know that the rationals are closed under $+$, $\cdot$, and additive/multiplicative inverses. In this problem, we explore what happens when we add, multiply, or take inverses of, irrational numbers. In each part, either prove or give a counterexample.
a. If $a \in \mathbb{R}\backslash\mathbb{Q}$ and $b \in \mathbb{Q}$, then $a + b \in \mathbb{R}\backslash\mathbb{Q}$.
b. If $a \in \mathbb{R}\backslash\mathbb{Q}$, then $-a \in \mathbb{R}\backslash\mathbb{Q}$.
c. If $a \in \mathbb{R}\backslash\mathbb{Q}$ and $b \in \mathbb{R}\backslash\mathbb{Q}$, then $a + b \in \mathbb{R}\backslash\mathbb{Q}$.
d. If $a \in \mathbb{R}\backslash\mathbb{Q}$ and $b \in \mathbb{Q}\backslash\{0\}$, then $ab \in \mathbb{R}\backslash\mathbb{Q}$.
e. If $a \in \mathbb{R}\backslash\mathbb{Q}$ and $b \in \mathbb{R}\backslash\mathbb{Q}$, then $ab \in \mathbb{R}\backslash\mathbb{Q}$.