Homework 12: Due Friday, December 14

Problem 1: Let $A \subseteq \mathbb{R}$.
a. Show that if f and g are bounded functions on A, then $f+g$ is a bounded function on A and that $||f+g|| \leq ||f|| + ||g||$.
b. Show that if f is a bounded function on A and $c \in \mathbb{R}$, then $c \cdot f$ is a bounded function on A and that $||c \cdot f|| = |c| \cdot ||f||$.
c. Give a counterexample to the following statement: If f and g are bounded functions on A, then $f \cdot g$ is a bounded function on A and $||f \cdot g|| = ||f|| \cdot ||g||$.

Problem 2: Let $A \subseteq \mathbb{R}$. Show that if $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on A, then $\langle f_n \rangle$ is uniformly convergent to the zero function on A.

Problem 3: Define a sequence of functions $\langle f_n \rangle$ on $\mathbb{R}$ by letting

$$f_n(x) = \frac{x}{1+nx^2}$$

for every $n \in \mathbb{N}^+$, and let $h \colon \mathbb{R} \to \mathbb{R}$ be the zero function. Show that $\langle f_n \rangle$ converges uniformly to h on $\mathbb{R}$.

Problem 4: Suppose that $a, b, c \in \mathbb{R}$ and $a < c < b$. Let $A = \{x \in \mathbb{R} : a < x < b \text{ and } x \neq c\}$. Suppose that $\langle f_n \rangle$ converges uniformly to h on A, and $\lim\limits_{x \to c} f_n(x)$ exists for all $n \in \mathbb{N}^+$. For each $n \in \mathbb{N}^+$, let $\ell_n = \lim\limits_{x \to c} f_n(x)$.
a. Show that $\langle \ell_n \rangle$ converges.
b. Show that $\lim\limits_{x \to c} h(x) = \lim\limits_{n \to \infty} \ell_n$.
Note: Part (b) is saying that, under these assumptions, we have $\lim\limits_{x \to c} \left(\lim\limits_{n \to \infty} f_n(x) \right) = \lim\limits_{n \to \infty} \left(\lim\limits_{x \to c} f_n(x) \right)$.

Problem 5: Find the radius of convergence of the following power series:

a. $\displaystyle\sum_{n=1}^{\infty} (-4)^n n x^n$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{n x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

Problem 6: Let $f \colon \mathbb{R} \backslash \{1, -2\} \to \mathbb{R}$ by letting

$$f(x) = \frac{3}{x^2 + x - 2}.$$

a. Find the Taylor series of f centered at 0, and find its radius of convergence.
b. Calculate $f^{(42)}(0)$.
Hint for a: Use partial fractions to decompose $\frac{3}{x^2+x-2}$ and then use geometric series.