Homework 11: Due Friday, November 30

Problem 1: Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$. Assume f satisfies each of the following:

1. $f(x) > 0$ for all $x \in \mathbb{R}$.
2. $\lim_{x \to \infty} f(x) = 0$.
3. $\lim_{x \to -\infty} f(x) = 0$.

Show that f attains a maximum value on $\mathbb{R}$, i.e. that there exists $z \in \mathbb{R}$ with $f(z) \geq f(x)$ for all $x \in \mathbb{R}$.

Problem 2: Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ are both differentiable on $\mathbb{R}$ and that $f \circ g = id_{\mathbb{R}}$, i.e. that $f(g(x)) = x$ for every $x \in \mathbb{R}$.
a. Show that $g'(a) \neq 0$ for every $a \in \mathbb{R}$.
b. Show that if $b \in \text{range}(g)$, then $f'(b) \neq 0$.

Problem 3: Let A be an interval. Suppose that f is continuous on A, is differentiable on $\text{int}(A)$, and that $f'(x) \neq 1$ for all $x \in A$. Show that f has at most one fixed point. That is, show that there is at most one $z \in A$ with $f(z) = z$.

Problem 4:
a. Suppose that $f, g\colon [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for all $x \in (a, b)$. Show that $f(b) < g(b)$.
b. Suppose that $f\colon [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and is twice differentiable on $(0, 1)$, i.e. that $f'(x)$ and $f''(x)$ exist for all $x \in (0, 1)$. Suppose also that $f(0) = f'(0) = 0$ and $f(1) = 5$. Show that there exists $c \in (0, 1)$ with $f''(c) \geq 10$. (In other words, if you start at rest and move 5 meters in 1 second, then at some point your acceleration must have been at least 10 meters per second^2).

Problem 5: Let $f, g\colon [a, b] \to \mathbb{R}$ be bounded. Suppose that f is integrable on $[a, b]$ and that $g(x) = f(x)$ for all $x \in [a, b)$, i.e. that g agrees with f at all points with the possible exception of b. Using Proposition 6.2.12, we know that g is integrable on $[a, b]$. Show that $\int_a^b g = \int_a^b f$.
Hint: It suffices to argue that for all $\varepsilon > 0$, we have both $\int_a^b g \leq (\int_a^b f) + \varepsilon$ and $\int_a^b f \leq (\int_a^b g) + \varepsilon$.
Aside: A similar argument works if g agrees with f at all but one point (anywhere in $[a, b]$). Using induction, the result then generalizes to the case where g agrees with f at all but finitely many points.