Homework 10: Due Friday, November 16

Problem 1: Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$. Show that the set $\{x \in \mathbb{R} : f(x) = 0\}$ is closed.

Problem 2: Let A be an interval and let $f\colon A \to \mathbb{R}$ be continuous on A. Suppose that f has the property that $f(x) \in \mathbb{Q}$ for all $x \in A$. Show that f is a constant function i.e. that there exists $q \in \mathbb{Q}$ such that $f(x) = q$ for all $x \in A$.

Problem 3: Suppose that $f\colon [0,1] \to [0,1]$ is continuous on $[0,1]$. Show that there exists $z \in [0,1]$ with $f(z) = z$.
Hint: Draw some pictures to get some intuition, and then make use of the Intermediate Value Theorem.

Problem 4: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Suppose that there exists $c \in \mathbb{R}$ with $0 < c < 1$ such that $|f(x) - f(y)| \leq c \cdot |x - y|$ for all $x, y \in \mathbb{R}$.
a. Show that f is uniformly continuous on $\mathbb{R}$.
b. Let $a_1 \in \mathbb{R}$. Define a sequence $\langle a_n \rangle$ recursively by starting with a_1, and letting $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}^+$. Show that $|a_{n+1} - a_n| \leq c^{n-1} \cdot |a_2 - a_1|$ for all $n \in \mathbb{N}^+$.
c. Let $a_1 \in \mathbb{R}$. Show that the sequence $\langle a_n \rangle$ defined in part (b) is a Cauchy sequence.
d. Let $a_1 \in \mathbb{R}$. Since the sequence $\langle a_n \rangle$ defined in part (b) is a Cauchy sequence, we know that it converges. Let $\ell = \lim\limits_{n \to \infty} a_n$. Show that $f(\ell) = \ell$.
Note: Problems 3 and 4d are examples of a collection of results known as "fixed-point theorems", in that they provide conditions that guarantee that a function has a point x with $f(x) = x$. Their generalizations to higher dimensional and more exotic spaces play an important role in mathematics.