Homework 1: Due Friday, September 7

Problem 1: Let $r \in \mathbb{R}$ with $r \neq 1$. Use induction to show that

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all $n \in \mathbb{N}$.

Problem 2: Use induction to show that $(1+x)^n \geq 1 + nx$ whenever $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geq -1$. Clearly explain where you are using the assumption that $x \geq -1$.

Problem 3: Define a sequence recursively by letting $a_1 = 0$ and letting $a_{n+1} = \frac{1}{3}(a_n + 1)$ for all $n \in \mathbb{N}$.
a. Show that $0 \leq a_n < 1$ for all $n \in \mathbb{N}$.
b. Show that $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

Problem 4: Let A, B, C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions.
a. Show that if $g \circ f$ is injective, then f is injective.
b. Show that if $g \circ f$ is surjective and g is injective, then f is surjective.

Problem 5: Let F be an ordered field. Using only the ordered field axioms, or the results in the course notes that are derived from them, show each of the following:
a. Show that if $a, b \in F$ and $a < b$, then $a < \frac{a+b}{2} < b$.
b. Show that if $a, b, c, d \in F$ and both $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
c. Show that if $a, b \in F$ and $0 < a < b$, then $a^2 < b^2$.
d. Show that if $a, b \in F$ and $a^2 = b^2$, then either $a = b$ or $a = -b$.