Problem Set 9

Problem 1: Show that

$$3a^4 - 4a^3b + b^4 \geq 0$$

for all $a, b \in \mathbb{R}$.

Problem 2: Show that it is never possible to partition a set of six consecutive integers into two subsets in such a way that the least common multiple of the number in one subset is equal to the least common multiple of the numbers in the other.

***Problem 3:** Determine if there exists an infinite sequence (a_n) of positive integers having all of the following properties:

- $a_m \nmid a_n$ whenever $m \neq n$.
- $\gcd(a_m, a_n) > 1$ for all m, n.
- $\gcd\{a_n : n \in \mathbb{N}\} = 1$.

***Problem 4:** Let $n \geq 2$ and let T_n be the number of nonempty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

***Problem 5:** Suppose that the sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

***Problem 6:** Is there a polynomial $P(x)$ with integer coefficients such that $P(10) = 400$, $P(14) = 440$, and $P(18) = 520$?