Problem Set 5

Problem 1: What is the greatest common divisor of the numbers in the set $\{16^n + 10n - 1 : n \in \mathbb{N}^+\}$?

Problem 2: Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2 B = B^2 A$, can $A^2 + B^2$ be invertible?

Problem 3: Two players play the following game. On their turn, each player selects an integer from 1 to 11 inclusive. The game stops when the sum of all the numbers played is at least 56, in which case the last player who selected a number is the winner. Determine a winning strategy for one of the players.

***Problem 4:** Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is never an integer when $n > 1$.

***Problem 5:** Evaluate

$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$$

where a and b are positive.

***Problem 6:** For each integer $n \geq 0$, let $S(n) = n - m^2$ where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

***Problem 7:** Determine all polynomials $P(x)$ with real coefficients such that $P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.