Problem Set 4

Problem 1: Show that if $n \in \mathbb{N}^+$, then $n^4 + 2n^3 + 2n^2 + 2n + 1$ is never a perfect square.

Problem 2: Show that every composite (i.e. non prime) integer greater than 1 is expressible as $xy+xz+yz+1$ with x, y, z positive integers.

Problem 3: Let k be a positive integer. The n^{th} derivative of $\frac{1}{x^k-1}$ has the form

$$\frac{P_n(x)}{(x^k-1)^{n+1}}$$

where $P_n(x)$ is a polynomial. Find $P_n(1)$.

***Problem 4:** Suppose that $f(x)$ is a differentiable function such that $f'(a-x) = f'(x)$ for all x satisfying $0 \le x \le a$. Evaluate $\int_0^a f(x)\, dx$.

***Problem 5:** Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

1. $f(f(n)) = n$ for all integers n
2. $f(f(n+2)+2) = n$ for all integers n
3. $f(0) = 1$

***Problem 6:** Find a three-digit number all of whose integral powers end with the same three digits as does the original number.

***Problem 7:** Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. For example, $23 = 9 + 8 + 6$.