Problem Set 3

Problem 1: Let $\mathbb{N}^+ = \{1, 2, 3, \dots\}$. Suppose that $f\colon \mathbb{N}^+ \to \mathbb{Z}$ has the following properties:

1. $f(2) = 2$
2. $f(mn) = f(m)f(n)$ for all m and n.
3. $f(m) > f(n)$ whenever $m > n$.

Show that $f(n) = n$ for all $n \in \mathbb{N}^+$.

Problem 2: For each $n \in \mathbb{N}^+$, evaluate the sum $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!$.

Problem 3: What is the smallest integer, which, when divided in turn by $2, 3, 4, \dots, 10$, leaves remainder $1, 2, 3, \dots, 9$ respectively?

***Problem 4:** A dart, thrown at random, hits the square target $[-1, 1] \times [-1, 1]$ (i.e. the set of points (x, y) with both $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$). Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$ where a, b, c, d are positive integers.

***Problem 5:** Let $f(x)$ be a polynomial with integer coefficients. Suppose that $f(a) = 7$ for (at least) four distinct integers a. Show that $f(a) \neq 14$ for all $a \in \mathbb{Z}$.

***Problem 6:** Define a_n recursively by letting $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$. Show that the integers $a_1, a_2, a_3, \dots$ are pairwise relatively prime.

***Problem 7:** An integer n, unknown to you, has been randomly chosen in the interval $[1, 2011]$ with uniform probability. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you must guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.