Problem Set 2

Problem 1: A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there always exist two whose midpoint is also a lattice point.

Problem 2: Suppose that $f\colon [a, b] \to \mathbb{R}$ is a continuous function. Suppose also that f takes on no value more than twice. Show that f must take on some value exactly once.

Problem 3:
a. Suppose that $q \in \mathbb{Q}$ is positive and satisfies $q + \frac{1}{q} \in \mathbb{Z}$. Show that $q = 1$.
b. Does there exist a positive $x \in \mathbb{R}$ with $x \neq 1$ such that $x + \frac{1}{x} \in \mathbb{Z}$? Explain.

*__Problem 4:__ Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

*__Problem 5:__ Find the last digit of 2^(3^(4^5)), i.e. of $2^{3^{4^5}}$. Be careful because exponentiation is not associative (for example, $2^{3^2} = 2^9 = 512$ while $(2^3)^2 = 8^2 = 64$).

*__Problem 6:__ Show that if $f(x)$ is a nonconstant polynomial with integer coefficients, then there are infinitely many m such that $f(m)$ is not prime.

*__Problem 7:__ Let n be a natural number. Find the sum of the digits over all numbers occurring in the list $1, 2, 3, \ldots, 10^n - 1$.