Problem Set 13

Problem 1: Show that there is only one $a \in \mathbb{R}$ for which there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x-y) = f(x) - f(y) + axy$ whenever $x, y \in \mathbb{R}$.

Problem 2: Let $n \in \mathbb{N}^+$. How many sequences $(a_1, a_2, \dots, a_k)$ with $1 \le a_i \le n$ are there such that $n \mid (a_1 + a_2 + \cdots + a_k)$?

***Problem 3:** A collection S of subsets of $\{1, 2, \dots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} many subsets in S.

***Problem 4:** Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\, dx$$

***Problem 5:** Let $P(x)$ be a polynomial of degree n such that $P(k) = \frac{1}{k}$ for $k \in \{1, 2, \dots, n+1\}$. Determine $P(n+2)$.

***Problem 6:** Let $a_1, a_2, \dots, a_{99}$ be a permutation of the numbers $1, 2, \dots, 99$. Show that there exists $m < n$ with $|a_m - m| = |a_n - n|$.

***Problem 7:** Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \ge 2$, what is the smallest possible number of elements in A_S?