Problem Set 12

Problem 1: Determine all triples $(x, y, z) \in \mathbb{Z}^3$ for which $x^3 + y^3 + z^3 = (x + y + z)^3$.

Problem 2: Form a directed graph as follows. Let the vertex set be the set of positive rational numbers $\mathbb{Q}^+$. For the directed edge set, each given $q \in \mathbb{Q}$ has two edges coming out it: One points to $q + 1$ and one points to $\frac{q}{q+1}$. In other words, we have

$$V = \mathbb{Q}^+ \qquad E = \{(a, b) \in V^2 : \text{Either } b = a + 1 \text{ or } b = \frac{a}{a+1}\}$$

Show that for any $q \in V$, there is a unique directed path from 1 to q.

Problem 3: For which nonnegative integers n and k is

$$(k + 1)^n + (k + 2)^n + (k + 3)^n + (k + 4)^n + (k + 5)^n$$

divisible by 5?

***Problem 4:** Let $n \in \mathbb{N}^+$ and let B_n be the set of all binary sequences of length n. Define addition on two elements of B_n to be component-wise where you let $1 + 1 = 0$. In group theory terms, we are thinking of B_n as the additive group $(\mathbb{Z}/2\mathbb{Z})^n$. Suppose that $f : B_n \to B_n$ is a function with the following two properties:

- $f(\mathbf{0}) = \mathbf{0}$ where $\mathbf{0}$ is the the sequence of all 0's.
- Whenever $a, b \in B_n$ differ in exactly k coordinates, we have that $f(a)$ and $f(b)$ differ in exactly k coordinates.

Show that if $a, b, c \in B_n$ with $a + b + c = \mathbf{0}$, then $f(a) + f(b) + f(c) = \mathbf{0}$.

***Problem 5:** Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n + 1)f_n(x + 1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

***Problem 6:** Determine the units (i.e. rightmost) digit of

$$\left\lfloor \frac{10^{20,000}}{10^{100} + 3} \right\rfloor$$

***Problem 7:** Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is finite?