Problem Set 11

Problem 1: Show that $\log_{10} 2$ is irrational.

Problem 2: Define a sequence by letting $a_1 = a_2 = a_3 = 1$ and letting

$$a_n = \frac{1 + a_{n-1}a_{n-2}}{a_{n-3}}$$

whenever $n \geq 4$. Show that $a_n \in \mathbb{Z}$ for all $n \in \mathbb{N}^+$.

*__Problem 3:__ Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that there exists a constant c such that $f(x) = cx$.

*__Problem 4:__ The squares of an 8×8 chessboard are filled with the numbers $\{1, 2, 3, \dots, 64\}$ in such a way that each number occurs in exactly one square. Prove that there are two adjacent squares (sharing a side) where the values differ by at least 5.

*__Problem 5:__ Given a partition π of the set $[9] = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that given any two partitions π and π' of $[9]$, there exist distinct $x, y \in [9]$ with $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

*__Problem 6:__ For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?