Problem Set 10

Problem 1: Fix $c \in \mathbb{R}$ with $0 < c < 1$. Define a sequence by letting $a_1 = 1 + c$ and $a_{n+1} = \frac{1}{a_n} + c$. Does there exists an $n \in \mathbb{N}^+$ with $a_n \leq 1$?

Problem 2: Two positive numbers with distinct first digits are multiplied together. Is it possible for the first digit of the product to fall strictly between the first digits of the two numbers?

Problem 3: Let $G(n)$ denote the closest integer to $\frac{(n+3)^2}{12}$. Explain why $G(n)$ is well-defined and show that $G(n) = n + G(n - 6)$ for all $n \in \mathbb{N}$ with $n \geq 7$.

*Problem 4: Find all real solutions to the equation $4x^2 - 40 \cdot \lfloor x \rfloor + 51 = 0$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

*Problem 5: Let $n \in \mathbb{N}$ with $n \geq 2$. Show that the polynomial $f(x) = x^n + 5x^{n-1} + 3$ is irreducible, i.e. show that $f(x)$ can not be expressed as the product of two polynomials of degree at least 1 with integer coefficients.

*Problem 6: Given a positive integer n, let

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all positive integers n such that

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2$$

*Problem 7: Let f_n be the nth Fibonacci number, so $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Show that for every $d \in \mathbb{N}^+$, there exists $n \in \mathbb{N}^+$ such that $d \mid f_n$.