Homework 5: Due Friday, February 25

Problem 1: Let $r \in \mathbb{R}$ with $r \neq 1$. Use induction to show that

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all $n \in \mathbb{N}$.

Problem 2: Let f_n be the sequence of Fibonacci numbers, i.e. $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Use induction to show that $\gcd(f_{n+1}, f_n) = 1$ for all $n \in \mathbb{N}$.

Problem 3: Let $a, b \in \mathbb{Z}$, and let $c \in \mathbb{N}^+$. Let $m = \gcd(a, b)$. In this problem we show that $\gcd(ca, cb) = cm$, i.e. that $\gcd(ca, cb) = c \cdot \gcd(a, b)$, by verifying the three defining properties:
a. Explain why $cm \geq 0$.
b. Show that cm is a common divisor of ca and cb.
c. Let $d \in \mathbb{Z}$ be an arbitrary common divisor of ca and cb. Show that $d \mid cm$.

Problem 4: Determine, with explanation, which numbers $n \in \mathbb{N}^+$ satisfy $d(n) = 14$.

Problem 5: Let $a, b \in \mathbb{N}^+$ and let $d = \gcd(a, b)$. Since d is a common divisor of a and b, we may fix $k, \ell \in \mathbb{N}$ with $a = kd$ and $b = \ell d$. Let $m = k\ell d$.
a. Show that $a \mid m$, that $b \mid m$, and that $dm = ab$.
b. Suppose that $n \in \mathbb{Z}$ is such that $a \mid n$ and $b \mid n$. Show that $m \mid n$.
Note: The number m is called the *least common multiple* of a and b and is written as $\operatorname{lcm}(a, b)$. Since $dm = ab$ from part (a), it follows that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$. Using this together with the Euclidean Algorithm, we can quickly compute least common multiples.

Problem 6: Define a function $\sigma \colon \mathbb{N}^+ \to \mathbb{N}^+$ by letting $\sigma(n)$ be the sum of all positive divisors of n. In other words, if $Div^+(n) = \{d_1, d_2, \ldots, d_m\}$, then

$$\sigma(n) = \sum_{i=1}^{m} d_i.$$

For example, $\sigma(6) = 1 + 2 + 3 + 6 = 12$.
a. Give a closed-form formula (i.e. a formula not involving a sum) for $\sigma(p^k)$ whenever $p \in \mathbb{N}^+$ is prime and $k \in \mathbb{N}^+$.
b. Show that $\sigma(ab) = \sigma(a) \cdot \sigma(b)$ whenever $a, b \in \mathbb{N}^+$ satisfy $\gcd(a, b) = 1$.
c. Use parts (a) and (b) to give a formula for $\sigma(n)$ in terms of the prime factorization of n.
Hint for (b): Suppose that $Div^+(a) = \{c_1, c_2, \ldots, c_k\}$ and $Div^+(b) = \{d_1, d_2, \ldots, d_m\}$. How can you determine $Div^+(ab)$ using the results of Section 3.3?