Homework 2: Due Friday, February 4

Problem 1: Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property holds, you should explain why. If a certain property fails, you should give a specific counterexample.
a. $A = \mathbb{Z}$ where $a \sim b$ means $a - b \neq 1$.
b. $A = \mathbb{Z}$ where $a \sim b$ means that both a and b are even.
c. $A = \mathbb{R}$ where $x \sim y$ means that $x^2 < y$.
d. $A = \mathcal{P}(\{1, 2, 3, 4, 5\})$ where $X \sim Y$ means that $X \cap Y \neq \emptyset$.
e. $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 3), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

Problem 2: Write down an example of an equivalence relation R on $\{1, 2, 3, 4\}$ having one equivalence class of size 1 and one equivalence class of size 3. Describe R explicitly by listing its elements.

Problem 3: Let $A = \mathbb{R} \times \mathbb{R}$ and consider the relation $\sim$ defined on A where $(a, b) \sim (c, d)$ means $a^2 + b^2 = c^2 + d^2$. One can check that $\sim$ is an equivalence relation on A (you do not need to do this). Describe the equivalence classes geometrically. Explain.

Problem 4: Let A be a set, and assume that R and S are both equivalence relations on A. Either prove or find a counterexample for each of the following:
a. $R \cup S$ is an equivalence relation on A.
b. $R \cap S$ is an equivalence relation on A.

Problem 5: A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

> "If $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity. This gives the reflexive property."

Now you know that their argument must be wrong because one of the examples in Problem 1 is symmetric and transitive but not reflexive. Pinpoint the error in your friend's argument. Be as explicit and descriptive as you can.

Problem 6: Show that $Div(a) = Div(-a)$ for all $a \in \mathbb{Z}$.