# Homework 14: Due Wednesday, May 11

**Problem 1:** In this problem, you will finish the proof of Proposition 5.7.3. Let $m = pq$ where $p, q \in \mathbb{N}^+$ are distinct primes. Let $k, \ell \in \mathbb{N}^+$ with $k\ell \equiv 1 \pmod{\varphi(m)}$. Show that $(a^k)^\ell \equiv a \pmod{m}$ for all $a \in \mathbb{Z}$.
*Hint:* It suffices to show that both $(a^k)^\ell \equiv a \pmod{p}$ and $(a^k)^\ell \equiv a \pmod{q}$.

**Problem 2:** Show that

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

for all $n \in \mathbb{N}^+$ by the interpreting the left-hand side as a certain Riemann sum.
*Note:* You can find a direct inductive proof of this fact in the proof of Proposition 2.1.8.

**Problem 3:** Let $n, k \in \mathbb{N}$ with $0 \leq k < n$.
a. Show that if $0 \leq k < \frac{n-1}{2}$, then $\binom{n}{k} < \binom{n}{k+1}$.
b. Show that if $\frac{n-1}{2} < k < n$, then $\binom{n}{k} > \binom{n}{k+1}$.
c. Show that if $k = \frac{n-1}{2}$, then $\binom{n}{k} = \binom{n}{k+1}$.
*Note:* This problem says that any row of Pascal's triangle is strictly increasing until the middle term(s), and then strictly decreasing afterwards. I suggest that you start by looking at the quotient $\binom{n}{k+1}/\binom{n}{k}$.

**Problem 4:** Show that there are arbitrarily large gaps in the primes, i.e. that for every $n \in \mathbb{N}^+$, there exist $n$ consecutive composite numbers.
*Hint:* Factorials are your friends.

**Problem 5:** Recall the definition of square-free from Problem 3 on Homework 6. Given $n \in \mathbb{N}^+$, show that $|\{k \in [n] : k \text{ is square-free}\}| \leq 2^{\pi(n)}$.

**Problem 6:** Using Problem 3 from Homework 6 together with the previous problem, show that

$$\pi(n) \geq \frac{\ln n}{2 \ln 2}$$

for all $n \in \mathbb{N}^+$.