Homework 9: Due Friday, March 12

Exercises

Exercise 1: For each of the following, either prove or find a counterexample.
a. Deleting a vertex of maximum degree in a finite graph G cannot increase the average degree.
b. Deleting a vertex of minimum degree in a finite graph G cannot decrease the average degree.

Exercise 2: Let T be the unique tree with vertex set $[8]$ whose Prüfer code is $4, 1, 1, 4, 3, 1$. Find the corresponding sequence $a_1, a_2, \dots, a_7$ and then draw T.

Exercise 3: Count the number of trees with vertex set $[11]$ where all of the following hold:

- $d(5) = 4$
- $d(1) = d(7) = 3$
- $d(4) = d(8) = 2$
- $d(v) = 1$ for all other vertices, i.e. all other vertices are leaves.

Exercise 4: Either prove or find a counterexample: Suppose that T is a minimum weight spanning tree of a connected weighted graph G. Let u and w be vertices of G. A u, w-path in T must have total weight less than or equal to the total weight of each u, w-path in G.

Problems

Problem 1: Given a graph G, we defined $\overline{G}$ in Problem 5 on Homework 8.
a. Let $n \geq 2$. Let G be a graph on n vertices with at least n edges. Show that G contains a cycle.
b. Give an example of graph on 4 vertices such that neither G nor $\overline{G}$ contains a cycle.
c. Show that if G is a graph on $n \geq 5$ vertices, then at least one of G or $\overline{G}$ contains a cycle.

Problem 2: Let T be a finite tree with n vertices. Let a_T be the average degree of the vertices (i.e. the result of summing the degrees of the vertices and dividing by n).
a. Show that $a_T < 2$.
b. Show that if T has a vertex of degree ℓ, then T has at least ℓ leaves.

Problem 3: Let T be a finite tree with at least two vertices and such that $d(v) \geq 3$ whenever v is adjacent to a leaf. Show that there exist two leaves u and w of T that share a common neighbor.
Hint: Start by considering a longest possible path in T.

Problem 4: Using Stirling numbers, count the number of trees with vertex set $[20]$ having exactly 6 leaves.

Problem 5: Let G be a finite connected graph that is not a tree. Show that G has at least 2 spanning trees.

Problem 6: Let G be a finite connected graph with at least 2 vertices. Show that there exist distinct vertices u and w such that both $G - u$ and $G - w$ are connected.
Hint: First think about the case where G is a tree.