Homework 8: Due Friday, March 5

Exercises

Exercise 1: Let G be a finite graph. Suppose that $v_1, v_2, \dots, v_n$ and $e_1, e_2, \dots, e_m$ are listings of the vertices and edges, and let B be the corresponding $n \times m$ incidence matrices. Show that the diagonal entry (i, i) of BB^T equals $d(v_i)$.
Note: Given a matrix B, the matrix B^T is the *transpose* of B, which is obtained by switching the rows and columns of B. So the (i, j) entry of B becomes the (j, i) entry of B^T.

Exercise 2: Suppose that G is a finite multigraph with exactly two vertices u and w of odd degree. Show that there exists a u, w-path in G.

Problems

Problem 1: Let G be a finite multigraph with n vertices and m edges. Let δ be the minimum degree of any vertex in G, and let Δ be the maximum degree of any vertex in G. Show that

$$\delta \leq \frac{2m}{n} \leq \Delta.$$

Problem 2: Let G be a connected graph. Suppose that e is an edge of G that is *not* contained in a cycle. Show that $G - e$ has exactly 2 connected components.

Problem 3: Show that the graph Q_n (defined in Homework 7) is connected for each $n \in \mathbb{N}^+$.

Problem 4: Fix $n \in \mathbb{N}^+$. Consider the graph G_n defined as follows. As in Q_n, let the vertex set V be the set of all sequences of 0's and 1's of length n. However, now let E be the set of all pairs $\{u, w\}$ such that u and w differ in exactly two coordinates.
a. Determine the number of edges in G_n.
b. Determine (with proof) the number of connected components in G_n.

Problem 5: Given a graph G, we define a new graph $\overline{G}$, called the complement of G, as follows. Let $V_{\overline{G}} = V_G$, i.e. the vertex set of $\overline{G}$ is the vertex set of G. Given two distinct vertices $u, w \in V_G$, we have that u and w are adjacent in $\overline{G}$ if and only if they are *not* adjacent in G. In other words, if $\mathcal{P}_2(V_G)$ is the set of all subsets of V_G of cardinality 2, then $E_{\overline{G}} = \mathcal{P}_2(V_G) \backslash E_G$.
a. Show that if G is a graph that is *not* connected, then $\overline{G}$ is connected.
b. Give an example of a connected graph G such that $\overline{G}$ is connected.

Problem 6: Let G be a graph. Suppose that no vertex of G is isolated (i.e. no vertex has degree 0) and that no induced subgraph of G has exactly two edges. Show that G is a complete graph, i.e. show that every pair of distinct vertices of G are adjacent.