Homework 7: Due Tuesday, March 2

Exercises

Exercise 1: Given a finite sequence of zeros and ones, define a *run* of ones to be a maximal consecutive subsequence of ones. For example, the sequence 11101100000100101100 has 5 runs of ones (and also 5 runs of zeros). Let $m, n, k \in \mathbb{N}^+$ with $k \leq m$ and $k \leq n$. Calculate the number of sequences of zeros and ones of length $m + n$ which have all three of the following properties:

- Have exactly m zeros and n ones.
- Starts with a one and ends with a zero.
- Have exactly k runs of ones.

Hint: How many runs of zeros must such a sequence have? Think about the lengths of the various runs and how they relate to compositions.

Exercise 2: Determine the number of solutions to

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where $x_i \in \mathbb{N}$ and $x_i \leq 6$ for all i.

Problems

Problem 1: Let $m, n \in \mathbb{N}^+$ with $m \leq n$. If σ is a permutation of $[n]$, then we say that $i \in [n]$ is a fixed point of σ if $\sigma(i) = i$. How many permutations of $[n]$ have exactly m fixed points?

Problem 2: In several cards games (bridge, spades, hearts, etc.), each player receives a 13-card hand from a standard 52-card deck.
a. How many such 13-card hands have at least one card of every suit? What percentage of all possible 13-card hands is this?
b. How many such 13-card hands have all four cards of some rank (e.g. all four queens)?

Problem 3: Consider all 10^{10} many ten-digit numbers where you allow leading zeros (so 0018345089 is one possibility). How many such numbers have the property that every odd digit occurs at least once?

Problem 4: Fix $n \in \mathbb{N}^+$. Consider the graph Q_n defined as follows. Let the vertex set V be the set of all sequences of 0's and 1's of length n (so for example, when $n = 3$, then one vertex is 010 and another is 111). Let E be the set of all pairs $\{u, v\}$ such that u and v differ in exactly one coordinate (so for example, when $n = 3$, there is edge with endpoints 001 and 101). The graph Q_n is called is called the n-cube.
a. Draw the graphs Q_1, Q_2, and Q_3.
b. Write down the adjacency matrix for Q_3 (clearly indicate the ordering of the vertices that you are using).
c. Let $U = \{0000, 0100, 1110, 1001, 1111\}$. Draw $Q_4[U]$.
d. Determine $d(u)$ for each $u \in Q_n$.
e. Determine the number of vertices and edges in Q_n.

Problem 5: Let G be a finite graph. Explain why the number of 1's in any adjacency matrix of G equals the number of 1's in any incidence matrix of G.

Problem 6: Let G be a finite graph with $|V| \geq 2$. Show that there exist $u, w \in V$ with $u \neq w$ such that $d(u) = d(w)$.