Homework 6: Due Friday, February 26

Exercises

Exercise 1: Recall that a *flush* in poker is a hand in which all five of your cards have the same suit. In class, we showed that there are $5,148$ many flushes (including straight flushes). Suppose that you are playing a game of poker in which each 2 is a "wild card". That is, you can take each 2 to represent any other card. For example, if you have three different hearts, the 2 of spades, and the 2 of diamonds, then this would be considered a flush because we can pretend that the two 2's are other hearts. In this situation, how many 5-card hands can be considered to be a flush? For this count, include any hand that could be viewed as a flush even if it could be viewed as a better hand (for example, if you have three 2's and two clubs, count that as a flush even though it can be viewed as four-of-a-kind).

Exercise 2:
a. Let $n \in \mathbb{N}^+$ and let $x \in \mathbb{R}$ with $x \geq 0$. Use the Binomial Theorem to show that $(1+x)^n \geq 1 + nx$.
b. Show that

$$1 \leq \sqrt[n]{2} \leq 1 + \frac{1}{n}$$

for all $n \in \mathbb{N}^+$.
Cultural Aside: Using the Squeeze Theorem, it follows that $\lim\limits_{n \to \infty} \sqrt[n]{2} = 1$.

Exercise 3: For all $k, n \in \mathbb{N}^+$ with $k \leq n$, we know that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$ since each side counts the number of ways of selecting a committee consisting of k people, including a distinguished president of the committee, from a group of n people.
a. Let $k, m, n \in \mathbb{N}^+$ with $m \leq k \leq n$. Give a combinatorial proof (i.e. argue that both sides count the same set) of the following:

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}.$$

This generalizes the above result (which is the special case where $m = 1$).
b. Let $m, n \in \mathbb{N}^+$ with $m \leq n$. Find a simple formula for:

$$\sum_{k=m}^{n} \binom{n}{k} \cdot \binom{k}{m}.$$

Exercise 4: Let $n, k \in \mathbb{N}^+$. Count the number of solutions to

$$x_1 + x_2 + \cdots + x_k \leq n$$

where each $x_i \in \mathbb{N}$. For example, if $n = 2$ and $k = 2$, then there are 6 solutions given by the following ordered pairs (x_1, x_2):

$$(0,0) \quad (0,1) \quad (1,0) \quad (0,2) \quad (1,1) \quad (2,0).$$

Your final answer should not involve any summations.

Problems

Problem 1: Snow White and the seven dwarfs (Bashful, Doc, Dopey, Grumpy, Happy, Sleepy and Sneezy) plan to eat dinner at a long rectangular table with four seats on each of two sides, and no seats on the two ends (so there there are four pairs of two seats opposite each other). Suppose that all seats are considered distinct.
a. Grumpy gets excessively grumpy if he is seated directly across the table from Sneezy. How many ways can the eight be seated so that those two are not across from each other?
b. Doc insists on sitting directly between Dopey and Sleepy to ensure they don't make fools of themselves. How many such seating arrangements are there?
c. How many arrangements satisfy both of the demands of part (a) and part (b)?

Problem 2: Let $n \in \mathbb{N}^+$. Determine (with explanation), the value of each of the following sums:
a.
$$\sum_{k=0}^{n} 2^k \cdot \binom{n}{k} = \binom{n}{0} + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 8 \cdot \binom{n}{3} + \cdots + 2^n \cdot \binom{n}{n}.$$
b.
$$\sum_{k=1}^{n} (-1)^{k-1} \cdot k \cdot \binom{n}{k} = \binom{n}{1} - 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} - 4 \cdot \binom{n}{4} + \cdots + (-1)^{n-1} \cdot n \cdot \binom{n}{n}.$$
c.
$$\sum_{k=2}^{n} k \cdot (k-1) \cdot \binom{n}{k} = 2 \cdot 1 \cdot \binom{n}{2} + 3 \cdot 2 \cdot \binom{n}{3} + \cdots + n \cdot (n-1) \cdot \binom{n}{n}.$$

Problem 3: Suppose that you have 12 identical apples and 1 orange.
a. In how many ways can you distribute the fruit to 4 distinct people?
b. In how many ways can you distribute the fruit to 4 distinct people in such a way that each person receives at least one piece of fruit?

Problem 4: Let $B(n, k)$ be the number of compositions of n into k parts. Using a direct combinatorial argument (so without using our formula), show that

$$B(n, k) = B(n-1, k-1) + B(n-1, k)$$

for all $n, k \in \mathbb{N}$ with $2 \leq k \leq n$.

Problem 5: Show that $S(n, 2) = 2^{n-1} - 1$ for all $n \geq 2$ in the following two ways:
a. By induction.
b. By a combinatorial argument.

Problem 6: Recall that $S(n, n-1) = \binom{n}{2}$ for all $n \geq 2$. Show that

$$S(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$$

for all $n \geq 3$.