Homework 5: Due Tuesday, February 23

Exercises

Exercise 1:
a. Recall that $\{0,1\}^*$ is the set of all finite sequences of 0's and 1's (of any finite length). Show that $\{0,1\}^*$ is countable.
b. Let S be the set of all infinite sequences of 0's and 1's (so an element of S looks like $11000101110\ldots$). Show that S is uncountable.

Exercise 2:
a. As in Exercise (1b), let S be the set of all infinite sequences of 0's and 1's. Show that there exists a bijection $f\colon \mathcal{P}(\mathbb{N}) \to S$.
b. Carefully explain why Exercise (1b) and part (a), taken together, imply that $\mathcal{P}(\mathbb{N})$ is uncountable.

Exercise 3: Using the digits 1 through 9 only (so exclude 0), how many 13 digits numbers are there in which no two consecutive digits are the same? Explain your reasoning.

Exercise 4: How many 6-letter "words" contain one of the letters A, B, C, D three times and each of the others once?

Problems

Problem 1: Show that if A and B are countable sets, then $A \times B$ is countable.

Problem 2: Show that the set $\mathbb{R}\backslash\mathbb{Q}$ of all irrational numbers is uncountable.

Problem 3: Suppose that you are creating a password using 26 letters, 10 numbers, and 15 special characters. How many such 10-character passwords are possible if they must have exactly 6 letters, 2 numbers, and 2 special characters?

Problem 4: How many ways are there to pick two cards from a standard 52-card deck such that the first card is a spade and the second is not an ace? In this problem, order matters. So if you pick the 3 of spades followed by the 7 of spades, this is different from the 7 of spades followed by the 3 of spades.

Problem 5: Suppose that a lottery draws 6 numbers from $[60] = \{1, 2, \ldots, 60\}$ without replacement and where order drawn doesn't matter. What percentage of possible lottery numbers have 3 evens and 3 odds?

Problem 6: A local pizza place has three different types of crust, five different meats, and seven different (non-meat) toppings. For a given pizza, you can pick any crust, at most 2 meats (so 0, 1, or 2 is possible) and at most 3 toppings (so 0, 1, 2, or 3 is possible). How many pizzas are possible?

Problem 7: In class, we talked about the number of paths starting at $(0, 0)$ and ending at (m, n) where each step was either one step north or one step east. How many such paths are there from $(0, 0)$ to $(12, 9)$ which do not go through the point $(5, 4)$? Think of needing to avoid that intersection because of construction.

Problem 8: How many 5-card poker hands have at least one card of every suit?