Homework 4: Due Tuesday, February 16

Exercises

Exercise 1: Show that if A is a finite set, then every injective function $f\colon A \to A$ is also surjective.
Note: Try to argue this as formally as possible using Fact 4.2.1.

Exercise 2:
a. For each $n \in \mathbb{N}^+$, give an example of a set $\mathcal{F} \subseteq \mathcal{P}(\{1,2,\dots,n\})$ with $|\mathcal{F}| = 2^{n-1}$ and such that $A \cap B \neq \emptyset$ whenever $A, B \in \mathcal{F}$. That is, $\mathcal{F}$ should be a collection of 2^{n-1} many subsets of $\{1,2,\dots,n\}$ with the property that any pair of sets from $\mathcal{F}$ have a common element.
b. Let $n \in \mathbb{N}^+$. Let $\mathcal{F} \subseteq \mathcal{P}(\{1,2,\dots,n\})$ be such that $|\mathcal{F}| > 2^{n-1}$, i.e. suppose that $\mathcal{F}$ is a collection of more than 2^{n-1} many subsets of $\mathcal{P}(\{1,2,\dots,n\})$. Show that there exists two elements of $\mathcal{F}$ that are disjoint.

Problems

Problem 1: Let A, B, C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Show that if $g \circ f$ is surjective and g is injective, then f is surjective.

Problem 2: A *lattice point* in the plane is a point of the form (a,b) where $a, b \in \mathbb{Z}$. For example, $(3,5)$ is a lattice point but $(\pi, 1)$ is not. Show that given any 5 lattice points in the plane, there exists two of the points whose midpoint is also a lattice point.

Problem 3:
a. For each $n \in \mathbb{N}^+$, give an example of a set $S \subseteq \{1,2,\dots,2n\}$ with $|S| = n$ such that $\gcd(a,b) > 1$ for all $a, b \in S$ with $a \neq b$.
b. Let $n \in \mathbb{N}^+$. Suppose that $A \subseteq \{1,2,\dots,2n\}$ and $|A| = n+1$. Show that there exists $a, b \in A$ with $a \neq b$ such that $\gcd(a,b) = 1$.

Problem 4: Let $n \in \mathbb{N}^+$. Given any $n+2$ many integers, show that it always possible to find two of them such that either their sum or their difference (or both) is divisible by $2n$.

Problem 5: Suppose that you have a group of 10 people and that the age of every person in the group is between 1 and 100 (inclusive). Suppose also that all of the ages are distinct, so there are not two people of the same age.
a. Show that there exist two nonempty distinct subsets A and B of people such that the sum of the ages of the people in A equals the sum of the ages of the people in B.
b. Show moreover that you can find A and B as in part a that are also *disjoint*, i.e. for which no person is in both A and B.
Example: Suppose that the ages of the people in the group are $3, 7, 13, 19, 24, 30, 38, 49, 63, 78$. One such example is $A = \{3, 13, 78\}$ and $B = \{7, 19, 30, 38\}$ since $3 + 13 + 78 = 94 = 7 + 19 + 30 + 38$.
Hint: How many possible nonempty subsets of people are there? What's the largest possible sum?