Homework 2: Due Tuesday, February 9

Exercises

Exercise 1: Let $A = \mathbb{R} \times \mathbb{R}$ and consider the relation $\sim$ defined on A where $(a, b) \sim (c, d)$ means $a^2 + b^2 = c^2 + d^2$. One can check that $\sim$ is an equivalence relation on A (you do not need to do this). Describe the equivalence classes geometrically. Explain.

Exercise 2: A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

> "If $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity. This gives the reflexive property."

Now you know that their argument must be wrong because one of the examples in Problem 5 on Homework 1 is symmetric and transitive, but not reflexive. Pinpoint the error in your friend's argument. Be as explicit and descriptive as you can.

Exercise 3: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid b$ and $a \nmid c$. Show that $a \nmid (b + c)$.

Exercise 4: Use induction to show that $6 \mid (2n^3 + 3n^2 + n)$ for all $n \in \mathbb{N}$.

Problems

Problem 1: Write down an example of an equivalence relation R on $\{1, 2, 3, 4\}$ having one equivalence class of size 1 and one equivalence class of size 3. Describe R explicitly by listing its elements.

Problem 2: Let A be a set, and assume that R and S are both equivalence relations on A. Either prove or find a counterexample for each of the following:
a. $R \cup S$ is an equivalence relation on A.
b. $R \cap S$ is an equivalence relation on A.

Problem 3: Show that $Div(a) = Div(-a)$ for all $a \in \mathbb{Z}$.

Problem 4: Define a sequence recursively by letting $a_0 = 42$ and letting

$$a_{n+1} = a_n^2 - 3a_n + 14$$

for all $n \in \mathbb{N}$. Show that $7 \mid a_n$ for all $n \in \mathbb{N}$.

Problem 5: Find a formula for

$$\sum_{k=1}^{n} (-1)^{k-1}(2k - 1) = 1 - 3 + 5 - 7 + 9 - \cdots + (-1)^{n-1}(2n - 1)$$

and prove that your formula is correct for all $n \in \mathbb{N}^+$.

Problem 6: Define a sequence by letting $a_0 = -1$, $a_1 = 7$, and $a_n = 2a_{n-1} + 4a_{n-2}$ for all $n \geq 2$. Show that $a_n \geq 3^n$ for all $n \in \mathbb{N}^+$.