Homework 10: Due Tuesday, March 16

Exercises

Exercise 1: Let G be a connected graph, and let P and Q be two paths of maximum length in G (so both P and Q have common length k, and there is no path of length greater than k). Show that P and Q have a common vertex.

Exercise 2: Let G be a graph. A *vertex cover* in G is a set $S \subseteq V$ such that every edge in G has at least one endpoint in S. Show that if M is a matching in G and S is a vertex cover in G, then $|M| \leq |S|$.

Problems

Problem 1: Recall the graph Q_n from Problem 4 on Homework 7. Show that Q_n is bipartite for each $n \in \mathbb{N}^+$.

Problem 2: Let G be the following graph. Let $V_G = \mathcal{P}_2([5])$, i.e. the vertices of G are the subsets of [5] of cardinality 2. For the edge set, let

$$E = \{\{u, w\} : u \text{ and } w \text{ are disjoint sets}\},$$

i.e. we include an edge with endpoints u and w exactly when u and w are 2-element subsets of [5] that do not share a common element. Determine, with justification, the value of $\chi(G)$.

Problem 3: Let T be a finite tree with $n \geq 2$ vertices.
a. Show that T is bipartite.
b. Show that given any proper 2-coloring of the vertices of T, there exists a leaf in the larger of the two color classes (and a leaf in each if both colors are used the same number of times).
Hint for (b): Think about the sum of the degrees in each color class.

Problem 4: Let $k \in \mathbb{N}^+$. A k-critical graph is a graph G such that $\chi(G) = k$ but $\chi(H) < k$ for any proper subgraph H of G, i.e. G has chromatic number k, but whenever we delete at least one vertex/edge, the chromatic number becomes strictly smaller. For example, every odd cycle is 3-critical (and in fact these are the only 3-critical graphs).
a. Show that K_n is an n-critical graph for each $n \in \mathbb{N}^+$.
b. Suppose that G is a k-critical graph. Show that $d(v) \geq k - 1$ for all $v \in V$.
Hint for (b): Suppose that $v \in V$ is such that $d(v) \leq k - 2$. Think about the graph $G - v$.

Problem 5: Given $n \in \mathbb{N}$ with $n \geq 3$, recall that C_n is the graph with vertex set $[n]$ where 1 and n are adjacent and also k and $k+1$ are adjacent whenever $1 \leq k \leq n-1$ (so C_n is just a cycle of length n).
a. Determine, with explanation, the size of a maximum matching in C_{300}.
b. Determine, with explanation, the size of a smallest possible maximal matching in C_{300}.

Problem 6: Let G be a graph. Let M be a maximum matching in G and suppose that M has k edges. Show that if L is a maximal matching in G, then L has at least $\frac{k}{2}$ many edges.