

Homework 1: Due Friday, February 5

Exercises

Exercise 1: Write out the following sets explicitly, and state their cardinalities:

- $\mathcal{P}(\{1, 2, 3\})$.
- $\mathcal{P}(\{1, \{2, 3\}\})$.
- $\mathcal{P}(\mathcal{P}(\{2\}))$.
- $\mathcal{P}(\emptyset)$ (be careful!).

Exercise 2: Give an example of three finite sets A_1, A_2, A_3 such that $A_1 \cap A_2 \cap A_3 = \emptyset$ but

$$|A_1 \cup A_2 \cup A_3| \neq |A_1| + |A_2| + |A_3|.$$

Exercise 3:

- Is $\mathbb{Z} \in \mathcal{P}(\mathbb{R})$? Is $\mathbb{Z} \subseteq \mathcal{P}(\mathbb{R})$? Explain.
- Does $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ for all sets A and B ? Explain.
- Does $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ for all sets A and B ? Explain.

Problems

Problem 1: Let A be a finite set, and let $n = |A|$. Let $D = \{(a, b) \in A^2 : a = b\}$.

- Write the set D parametrically.
- What is $|A^2 \setminus D|$? Explain.

Problem 2: Given two sets A and B , we define

$$A \triangle B = \{x : x \text{ is an element of exactly one of } A \text{ or } B\},$$

and we call this set the *symmetric difference* of A and B .

- Determine $\{1, 3, 8, 9\} \triangle \{2, 3, 4, 7, 8\}$.
 - Determine $\{1, 2, 3\} \triangle \{1, \{2, 3\}\}$.
 - What are the smallest 9 elements of the set $\{2n : n \in \mathbb{N}\} \triangle \{3n : n \in \mathbb{N}\}$?
 - Make a conjecture about how to write $\{2n : n \in \mathbb{N}\} \triangle \{3n : n \in \mathbb{N}\}$ as the union of 3 pairwise disjoint sets (no need to prove this conjecture, but do write the 3 sets parametrically).
 - Show that if A and B are finite sets, then $|A \triangle B| = |A \setminus B| + |B \setminus A|$.
 - Show that if A and B are finite sets, then $|A \triangle B| = |A \cup B| - |A \cap B|$.
- Note:* For parts (e) and (f), appeal to the fundamental counting rules from class or Section 1.2 of the notes.

Problem 3: Write a careful proof showing that $\{6n^2 + 14 : n \in \mathbb{Z}\} \subseteq \{2n - 8 : n \in \mathbb{Z}\}$.

Problem 4: Write a careful double containment proof showing that $\{\sqrt{5x^2 + 9} : x \in \mathbb{R}\} = \{x \in \mathbb{R} : x \geq 3\}$.
Recall: Given $x \in \mathbb{R}$ with $x \geq 0$, we define \sqrt{x} to be the unique *nonnegative* $y \in \mathbb{R}$ with $y^2 = x$.

Problem 5: Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property holds, you should explain why. If a certain property fails, you should give a specific counterexample.

- a. $A = \mathbb{Z}$ where $a \sim b$ means $a - b \neq 1$.
- b. $A = \mathbb{Z}$ where $a \sim b$ means that both a and b are even.
- c. $A = \mathbb{R}$ where $x \sim y$ means that $x^2 < y$.
- d. $A = \mathcal{P}(\{1, 2, 3, 4, 5\})$ where $X \sim Y$ means that $X \cap Y \neq \emptyset$.
- e. $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 3), (2, 3), (3, 1), (3, 2), (3, 3)\}$.