Combinatorics and Graph Theory

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Chapter 1

Introduction

1.1 Sets, Set Construction, and Subsets

Sets and Set Construction

We begin by reviewing the fundamental structure that mathematicians use to package objects together.

Definition 1.1.1. *A* set *is a collection of elements without regard to repetition or order.*

Intuitively, a set is a box where the only thing that matters are the objects that are inside it, and furthermore the box does not have more than one of any given object. We use { and } as delimiters for sets. For example, $\{3, 5\}$ is a set with two elements. Since all that matters are the elements, we define two sets to be equal if they have the same elements, regardless of how the sets themselves are defined or described.

Definition 1.1.2. *Given two sets A and B, we say that $A = B$ if A and B have exactly the same elements.*

Since only the actual elements matter, not their order, we have $\{3, 7\} = \{7, 3\}$ and $\{1, 2, 3\} = \{3, 1, 2\}$. Also, although we typically would not even write something like $\{2, 5, 5\}$, if we choose to do so then we would have $\{2, 5, 5\} = \{2, 5\}$ because both have the same elements, namely 2 and 5.

Notation 1.1.3. *Given an object x and a set A, we write $x \in A$ to mean that x is an element of A, and we write $x \notin A$ to mean that x is not an element of A.*

For example, we have $2 \in \{2, 5\}$ and $3 \notin \{2, 5\}$. Since sets are mathematical objects, they may be elements of other sets. For example, we can form the set $S = \{1, \{2, 3\}\}$. Notice that we have $1 \in S$ and $\{2, 3\} \in S$, but $2 \notin S$ and $3 \notin S$. As a result, S has only 2 elements, namely 1 and $\{2, 3\}$. Thinking of a set as a box, one element of S is the number 1, and the other is a different box (which happens to have the two elements 2 and 3 inside it).

The empty set is the unique set with no elements. We can write it as $\{\}$, but instead we typically denote it by $\emptyset$. There is only *one* empty set, because if both A and B have no elements, then they have exactly the same elements for vacuous reasons, and hence $A = B$. Notice that $\{\emptyset\}$ does not equal $\emptyset$. After all, $\{\emptyset\}$ has one element! You can think of $\{\emptyset\}$ as a box that has one empty box inside it.

Notice that sets can be either finite or infinite. At this point, our standard examples of infinite sets are the various universes of numbers:

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.
- $\mathbb{N}^+ = \{1, 2, 3, \dots\}$.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

- $\mathbb{Q}$ is the set of rational numbers.
- $\mathbb{R}$ is the set of real numbers.

Beyond these fundamental sets, there are various ways to define new sets. In some cases, we can simply list the elements as we did above. Although this often works for small finite sets, it is almost never a good idea to list the elements of a set with 20 or more elements, and it rarely works for infinite sets (unless there is an obvious pattern like $\{5, 10, 15, 20, \dots\}$). One of the standard ways to define a set S is to carve it out of some bigger set A by describing a certain property that may or may not be satisfied by an element of A. For example, we could define

$$S = \{n \in \mathbb{N} : 5 < n < 13\}.$$

We read this line by saying that S is defined to be the set of all $n \in \mathbb{N}$ such that $5 < n < 13$. Thus, in this case, we are taking $A = \mathbb{N}$, and forming a set S by carving out those elements of A that satisfy the condition that $5 < n < 13$. In other words, think about going through each of element n, checking if $5 < n < 13$ is a true statement, and collecting those $n \in \mathbb{N}$ that make it true into a set that we call S. In more simple terms, we can also describe S as follows:

$$S = \{6, 7, 8, 9, 10, 11, 12\}.$$

It is important that we put the "$\mathbb{N}$" in the above description, because if we wrote $\{n : 5 < n < 13\}$ then it would be unclear what n we should consider. For example, should $\frac{11}{2}$ be in this set? How about $\sqrt{17}$? Sometimes the "universe" of numbers (or other mathematical objects) that we are working within is clear, but typically it is best to write the global set that we are picking elements from in order to avoid such ambiguity. Notice that when we define a set, there is no guarantee that it has any elements. For example, $\{q \in \mathbb{N} : q^2 = 2\} = \emptyset$ because $\sqrt{2}$ is irrational (we will prove this, and much more general facts, later). Keep in mind that we can also use words in our description of sets, such as $\{n \in \mathbb{N} : n \text{ is an even prime}\}$. As mentioned above, two sets that have quite different descriptions can be equal. For example, we have

$$\{n \in \mathbb{N} : n \text{ is an even prime}\} = \{n \in \mathbb{N} : 3 < n^2 < 8\}$$

because both sets equal $\{2\}$. Always remember the structure of sets formed in this way. We write

$$\{x \in A : \mathsf{P}(x)\}$$

where A is a known set and $\mathsf{P}(x)$ is a "property" such that given a particular $y \in A$, the statement $\mathsf{P}(y)$ is either true or false.

Another way to describe a set is through a "parametric" description. Rather than carving out a certain subset of a given set by describing a property that the elements must satisfy, we can instead form all the elements one obtains by varying a value through a particular set. For example, consider the following description of a set:

$$S = \{3x^2 + 1 : x \in \mathbb{R}\}.$$

Although the notation looks quite similar to the above (in both case we have curly braces, with a : in the middle), this set is described differently. Notice that instead of having a set that elements are coming from on the left of the colon, we now have a set that elements are coming from on the right. Furthermore, we now have a formula on the left rather than a property on the right. The difference is that for a property, when we plug in an element from the given set, we either obtain a true or false value, but that isn't the case for a formula like $3x^2 + 1$. The idea here is that instead of carving out a subset of $\mathbb{R}$ by using a property (i.e. taking those elements that make the property *true*), we let x vary through all real numbers, plug each of these real numbers x into $3x^2 + 1$, and form the set of all possible outputs. For example, we have $4 \in S$ because $4 = 3 \cdot 1^2 + 1$. In other words, when $x = 1$, the left hand side gives the value 4, so we should put $4 \in S$. Notice also that $4 = 3 \cdot (-1)^2 + 1$, so we can also see that $4 \in S$ because of the "witness" -1. Of

course, we are forming a set, so we do not repeat the number 4. We also have $1 \in S$ because $1 = 3 \cdot 0^2 + 1$, and we have $76 \in S$ because $76 = 3 \cdot 5^2 + 1$. Notice also that $7 \in S$ because $7 = 3 \cdot (\sqrt{2})^2 + 1$.

In a general parametric set description, we will have a set A and a *function* $f(x)$ that allows inputs from A, and we write

$$\{f(x) : x \in A\}$$

for the set of all possible outputs of the function as we vary the inputs through the set A. We will discuss the general definition of a function in the next section, but for the moment you can think of them as given by formulas.

Now it is possible and indeed straightforward to turn any parametric description of a set into one where we carve out a subset by a property. In our case of $S = \{3x^2 + 1 : x \in \mathbb{R}\}$ above, we can alternatively write it as

$$S = \{y \in \mathbb{R} : \text{There exists } x \in \mathbb{R} \text{ with } y = 3x^2 + 1\}.$$

Notice how we flipped the way we described the set by introducing a "there exists" quantifier in order to form a property. This is always possible for a parametric description. For example, we have

$$\{5n + 4 : n \in \mathbb{N}\} = \{m \in \mathbb{N} : \text{There exists } n \in \mathbb{N} \text{ with } m = 5n + 4\}.$$

Thus, these parametric descriptions are not essentially new ways to describe sets, but they can often be more concise and clear.

By the way, we can use multiple parameters in our description. For example, consider the set

$$S = \{18m + 33n : m, n \in \mathbb{Z}\}.$$

Now we are simply letting m and n vary through all possible values in $\mathbb{Z}$ and collecting all of the values $18m + 33n$ that result. For example, we have $15 \in S$ because $15 = 18 \cdot (-1) + 33 \cdot 1$. We also have $102 \in S$ because $102 = 18 \cdot 2 + 33 \cdot 2$. Notice that we are varying m and n independently, so they might take different values, or the same value (as in the case of $m = n = 2$). Don't be fooled by the fact that we used different letters! As above, we can flip this description around by writing

$$S = \{k \in \mathbb{Z} : \text{There exists } m, n \in \mathbb{Z} \text{ with } k = 18m + 33n\}.$$

Subsets and Set Equality

Definition 1.1.4. *Given two sets A and B, we write $A \subseteq B$ to mean that every element of A is an element of B. More formally, $A \subseteq B$ means that for all x, if $x \in A$, then $x \in B$.*

Written more succinctly, $A \subseteq B$ means that for all $a \in A$, we have that $a \in B$. To prove that $A \subseteq B$, one takes a completely arbitrary $a \in A$, and argues that $a \in B$. For example, let $A = \{6n : n \in \mathbb{Z}\}$ and let $B = \{2n : n \in \mathbb{Z}\}$. Since both of these sets are infinite, we can't show that $A \subseteq B$ by taking each element of A in turn and showing that it is an element of B. Instead, we take an *arbitrary* $a \in A$, and show that $a \in B$. Here's the proof.

Proposition 1.1.5. *Let $A = \{6n : n \in \mathbb{Z}\}$ and $B = \{2n : n \in \mathbb{Z}\}$. We have $A \subseteq B$.*

Proof. Let $a \in A$ be arbitrary. By definition of A, this means that we can fix an $m \in \mathbb{Z}$ with $a = 6m$. Notice then that $a = 2 \cdot (3m)$. Since $3m \in \mathbb{Z}$, it follows that $a \in B$. Since $a \in A$ we arbitrary, we conclude that $A \subseteq B$. □

As usual, pause to make sure that you understand the logic of the argument above. First, we took an arbitrary element a from the set A. Now since $A = \{6n : n \in \mathbb{Z}\}$ and this is a parametric description with an implicit "there exists" quantifier, there must be one fixed integer value of n that puts a into the set A.

In our proof, we chose to call that one fixed integer m. Now in order to show that $a \in B$, we need to exhibit a $k \in \mathbb{Z}$ with $a = 2k$. In order to do this, we hope to manipulate $a = 6m$ to introduce a 2, and ensure that the element we are multiplying by 2 is an integer.

What would go wrong if we tried to prove that $B \subseteq A$? Let's try it. Let $b \in B$ be arbitrary. Since $b \in B$, we can fix $m \in \mathbb{Z}$ with $b = 2m$. Now our goal is to try to prove that we can find an $n \in \mathbb{Z}$ with $b = 6n$. It's not obvious how to obtain a 6 from that 2, but we can try to force a 6 in the following way. Since $b = 2m$ and $2 = \frac{6}{3}$, we can write $b = 6 \cdot \frac{m}{3}$. We have indeed found a number n such that $b = 6n$, but we have not checked that this n is an integer. In general, dividing an integer by 3 does not result in an integer, so this argument currently has a hole in it.

Although that argument has a problem, we can not immediately conclude that $B \not\subseteq A$. Our failure to find an argument does not mean that an argument does not exist. So how can we show that $B \not\subseteq A$? All that we need to do is find just *one example* of an element of B that is not an element of A (because the negation of the "for all" statement $A \subseteq B$ is a "there exists" statement). We choose 2 as our example. However, we need to convince everybody that this choice works. So let's do it! First, notice that $2 = 2 \cdot 1$, so $2 \in B$ because $1 \in \mathbb{Z}$. We now need to show that $2 \notin A$, and we'll do this using a proof by contradiction. Suppose instead that $2 \in A$. Then, by definition, we can fix an $m \in \mathbb{Z}$ with $2 = 6m$. We then have that $m = \frac{2}{6} = \frac{1}{3}$. However, this a contradiction because $\frac{1}{3} \notin \mathbb{Z}$. Since our assumption that $2 \in A$ led to a contradiction, we conclude that $2 \notin A$. We found an example of an element that is in B but not in A, so we conclude that $B \not\subseteq A$.

Recall that two sets A and B are defined to be equal if they have the same elements. Therefore, we have $A = B$ exactly when both $A \subseteq B$ and $B \subseteq A$ are true. Thus, given two sets A and B, we can prove that $A = B$ by performing two proofs like the one above. Such a strategy is called a *double containment* proof. We give an example of such an argument now.

Proposition 1.1.6. *Let $A = \{7n - 3 : n \in \mathbb{Z}\}$ and $B = \{7n + 11 : n \in \mathbb{Z}\}$. We have $A = B$.*

Proof. We prove that $A = B$ by showing that both $A \subseteq B$ and also that $B \subseteq A$.

- We first show that $A \subseteq B$. Let $a \in A$ be arbitrary. By definition of A, we can fix an $m \in \mathbb{Z}$ with $a = 7m - 3$. Notice that

$$\begin{aligned} a &= 7m - 3 \\ &= 7m - 14 + 11 \\ &= 7(m - 2) + 11. \end{aligned}$$

  Now $m - 2 \in \mathbb{Z}$ because $m \in \mathbb{Z}$, so it follows that $a \in B$. Since $a \in A$ was arbitrary, we conclude that $A \subseteq B$.

- We now show that $B \subseteq A$. Let $b \in B$ be arbitrary. By definition of B, we can fix an $m \in \mathbb{Z}$ with $b = 7m + 11$. Notice that

$$\begin{aligned} b &= 7m + 11 \\ &= 7m + 14 - 3 \\ &= 7(m + 2) - 3. \end{aligned}$$

  Now $m + 2 \in \mathbb{Z}$ because $m \in \mathbb{Z}$, so it follows that $b \in A$. Since $b \in B$ was arbitrary, we conclude that $B \subseteq A$.

We have shown that both $A \subseteq B$ and $B \subseteq A$ are true, so it follows that $A = B$. □

Here is a more interesting example. Consider the set

$$S = \{9m + 15n : m, n \in \mathbb{Z}\}.$$

For example, we have $9 \in S$ because $9 = 9 \cdot 1 + 15 \cdot 0$. We also have $3 \in S$ because $3 = 9 \cdot 2 + 15 \cdot (-1)$ (or alternatively because $3 = 9 \cdot (-3) + 15 \cdot 2$). We can always generate new values of S by simply plugging in values for m and n, but is there another way to describe the elements of S in an easier way? We now show that an integer is in S exactly when it is a multiple of 3.

Proposition 1.1.7. *We have* $\{9m + 15n : m, n \in \mathbb{Z}\} = \{3m : m \in \mathbb{Z}\}$.

Proof. We give a double containment proof.

- We first show that $\{9m + 15n : m, n \in \mathbb{Z}\} \subseteq \{3m : m \in \mathbb{Z}\}$. Let $a \in \{9m + 15n : m, n \in \mathbb{Z}\}$ be arbitrary. By definition, we can fix $k, \ell \in \mathbb{Z}$ with $a = 9k + 15\ell$. Notice that

$$\begin{aligned} a &= 9k + 15\ell \\ &= 3 \cdot (3k + 5\ell). \end{aligned}$$

  Now $3k+5\ell \in \mathbb{Z}$ because $k, \ell \in \mathbb{Z}$, so it follows that $a \in \{3m : m \in \mathbb{Z}\}$. Since $a \in \{9m+15n : m, n \in \mathbb{Z}\}$ was arbitrary, we conclude that $\{9m + 15n : m, n \in \mathbb{Z}\} \subseteq \{3m : m \in \mathbb{Z}\}$.

- We now show that $\{3m : m \in \mathbb{Z}\} \subseteq \{9m + 15n : m, n \in \mathbb{Z}\}$. Let $a \in \{3m : m \in \mathbb{Z}\}$ be arbitrary. By definition, we can fix $k \in \mathbb{Z}$ with $a = 3k$. Notice that

$$\begin{aligned} a &= 3k \\ &= (9 \cdot (-3) + 15 \cdot 2) \cdot k \\ &= 9 \cdot (-3k) + 15 \cdot 2k. \end{aligned}$$

  Now $-3k, 2k \in \mathbb{Z}$ because $k \in \mathbb{Z}$, so it follows that $a \in \{9m+15n : m, n \in \mathbb{Z}\}$. Since $a \in \{3m : m \in \mathbb{Z}\}$ was arbitrary, we conclude that $\{3m : m \in \mathbb{Z}\} \subseteq \{9m + 15n : m, n \in \mathbb{Z}\}$.

We have shown that both $\{9m+15n : m, n \in \mathbb{Z}\} \subseteq \{3m : n \in \mathbb{Z}\}$ and $\{3m : m \in \mathbb{Z}\} \subseteq \{9m+15n : m, n \in \mathbb{Z}\}$ are true, so it follows that $\{9m + 15n : m, n \in \mathbb{Z}\} = \{3m : m \in \mathbb{Z}\}$. □

Ordered Pairs and Sequences

In contrast to sets, we define *ordered pairs* in such a way that order and repetition *do* matter. We denote an ordered pair using normal parentheses rather than curly braces. For example, we let $(2, 5)$ be the ordered pair whose first element is 2 and whose second element is 5. Notice that we have $(2, 5) \neq (5, 2)$ despite the fact that $\{2, 5\} = \{5, 2\}$. Make sure to keep a clear distinction between the ordered pair $(2, 5)$ and the set $\{2, 5\}$. We *do* allow the possibility of an ordered pair such as $(2, 2)$, and here the repetition of 2's is meaningful. Furthermore, we do not use $\in$ in ordered pairs, so we would **not** write $2 \in (2, 5)$. We'll talk about ways to refer to the two elements of an ordered pair later.

We can generalize ordered pairs to the possibility of having more than 2 elements. In this case, we have an ordered list of n elements, like $(5, 4, 5, -2)$. We call such an object an *n-tuple*, a *list* with n elements, or a finite *sequence* of length n. Thus, for example, we could call $(5, 4, 5, -2)$ a 4-tuple. It is also possible to have infinite sequences (i.e. infinite lists), but we will wait to discuss these until the time comes.

Operations on Sets and Sequences

Aside from listing elements, carving out subsets of a given set using a given property, and giving a parametric description (which as mentioned above is just a special case of the previous type), there are other ways to build sets.

Definition 1.1.8. *Given two sets A and B, we define $A \cup B$ to be the set consisting of those elements that are in A or B (or both). In other words, we define*

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

We call this set the union *of A and B.*

Here, as in mathematics generally, we use *or* to mean "inclusive or". In other words, if x is an element of both A and B, then we still put x into $A \cup B$. Here are a few examples (we leave the proofs of the latter results until we have more theory):

- $\{1, 2, 7\} \cup \{4, 9\} = \{1, 2, 4, 7, 9\}$.
- $\{1, 2, 3\} \cup \{2, 3, 5\} = \{1, 2, 3, 5\}$.
- $\{2n : n \in \mathbb{N}\} \cup \{2n + 1 : n \in \mathbb{N}\} = \mathbb{N}$.
- $\{2n : n \in \mathbb{N}^+\} \cup \{2n + 1 : n \in \mathbb{N}^+\} = \{2, 3, 4, \dots\}$.
- $\{2n : n \in \mathbb{N}^+\} \cup \{2n - 1 : n \in \mathbb{N}^+\} = \{1, 2, 3, 4, \dots\} = \mathbb{N}^+$.
- $A \cup \emptyset = A$ for every set A.

Definition 1.1.9. *Given two sets A and B, we define $A \cap B$ to be the set consisting of those elements that are in both of A and B. In other words, we define*

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We call this set the intersection *of A and B.*

Here are a few examples (again we leave some proofs until later):

- $\{1, 2, 7\} \cap \{4, 9\} = \emptyset$.
- $\{1, 2, 3\} \cap \{2, 3, 5\} = \{2, 3\}$.
- $\{1, \{2, 3\}\} \cap \{1, 2, 3\} = \{1\}$.
- $\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} = \{6n : n \in \mathbb{Z}\}$.
- $\{3n + 1 : n \in \mathbb{N}^+\} \cap \{3n + 2 : n \in \mathbb{N}^+\} = \emptyset$.
- $A \cap \emptyset = \emptyset$ for every set A.

Definition 1.1.10. *Given two sets A and B, we define $A \backslash B$ to be the set consisting of those elements that are in A, but not in B. In other words, we define*

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

We call this set the (relative) complement *of B (in A).*

In many cases where we consider $A\backslash B$, we will have that $B \subseteq A$, but we will occasionally use it even when $B \not\subseteq A$. Here are a few examples:

- $\{5,6,7,8,9\}\backslash\{5,6,8\} = \{7,9\}$.
- $\{1,2,7\}\backslash\{4,9\} = \{1,2,7\}$.
- $\{1,2,3\}\backslash\{2,3,5\} = \{1\}$.
- $\{2n : n \in \mathbb{Z}\}\backslash\{4n : n \in \mathbb{Z}\} = \{4n+2 : n \in \mathbb{Z}\}$.
- $A\backslash\emptyset = A$ for every set A.
- $A\backslash A = \emptyset$ for every set A.

Definition 1.1.11. *Given two sets A and B, we let $A \times B$ be the set of all ordered pairs (a,b) such that $a \in A$ and $b \in B$, and we call this set the* Cartesian product *of A and B.*

For example, we have

$$\{1,2,3\} \times \{6,8\} = \{(1,6),(1,8),(2,6),(2,8),(3,6),(3,8)\}$$

and

$$\mathbb{N} \times \mathbb{N} = \{(0,0),(0,1),(1,0),(2,0),\dots,(4,7),\dots\}.$$

Notice that elements of $\mathbb{R} \times \mathbb{R}$ correspond to points in the plane.

We can also generalize the concept of a Cartesian product to more than 2 sets. If we are given n sets $A_1, A_2, \dots, A_n$, we let $A_1 \times A_2 \times \cdots \times A_n$ be the set of all n-tuples $(a_1, a_2, \dots, a_n)$ such that $a_i \in A_i$ for each i. For example, we have

$$\{1,2\} \times \{3\} \times \{4,5\} = \{(1,3,4),(1,3,5),(2,3,4),(2,3,5)\}$$

In the special case when $A_1, A_2, \dots, A_n$ are all the same set A, we use the notation A^n to denote the set $A \times A \times \cdots \times A$ (where we have n copies of A). Thus, A^n is the set of all finite sequences of elements of A of length n. For example, $\{0,1\}^n$ is the set of all finite sequences of 0's and 1's of length n. Notice that this notation fits in with the notation $\mathbb{R}^n$ that we are used to in Calculus and Linear Algebra.

Definition 1.1.12. *Given a set A, we let $\mathcal{P}(A)$ be the set of all subsets of A, and we call $\mathcal{P}(A)$ the* power set *of A.*

For example, we have

$$\mathcal{P}(\{1,2\}) = \{\emptyset,\{1\},\{2\},\{1,2\}\}$$

and

$$\mathcal{P}(\{4,5,7\}) = \{\emptyset,\{4\},\{5\},\{7\},\{4,5\},\{4,7\},\{5,7\},\{4,5,7\}\}.$$

Notice that it is can be tricky to write out the power set of even small finite sets. We'll see ways to both generate and count the number of elements of $\mathcal{P}(A)$ for a given set A a bit later.

Definition 1.1.13. *Given a set A, we let A^* be the set of all finite sequences of elements of A of any length, including the empty sequence (the unique sequence of length 0).*

Thus, for example, the set $\{0,1\}^*$ is the set of all finite sequences of 0's and 1's. If we use λ to denote the empty sequence and write things like 010 in place of the more precise $(0,1,0)$, then we have

$$\{0,1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, \dots\}.$$

Notice that if $A \neq \emptyset$, then A^* is an infinite set.

Definition 1.1.14. *Given two finite sequences σ and τ, we let $\sigma\tau$ be the* concatenation *of σ and τ, i.e. if $\sigma = (a_1, a_2, \dots, a_m)$ and $\tau = (b_1, b_2, \dots, b_n)$, then $\sigma\tau = (a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n)$.*

1.2 The Cardinality of Sets

We will spend a significant amount of time trying to count the number of elements in certain sets. For now, we will study some simple properties that will eventually become extremely useful when employed in clever ways.

Definition 1.2.1. *Given a set A, we let $|A|$ be the number of elements of A, and we call $|A|$ the cardinality of A. If A is infinite, then we write $|A| = \infty$.*

Of course, if we list the elements of a set A, then it's usually quite easy to determine $|A|$. For example, we trivially have $|\{1, \sqrt{2}, \frac{5}{2}, 18\}| = 4$. However, it can be very hard to determine the cardinality of a set. For example, consider the set

$$A = \{(x, y) \in \mathbb{Z}^2 : y^2 = x^3 - 1\}.$$

Determining the elements of A is not easy. It is easy to see that $(1, 0) \in A$, but it is not clear whether there are any other elements. Using some sophisticated number theory, it is possible to show that $A = \{(1, 0)\}$, and hence $|A| = 1$.

We start with one of the most basic, yet important, rules about the cardinality of sets.

Definition 1.2.2. *We say that two sets A and B are disjoint if $A \cap B = \emptyset$.*

Fact 1.2.3 (Sum Rule)**.** *If A and B are finite disjoint sets, then $|A \cup B| = |A| + |B|$.*

We won't give a formal proof of this fact, because it is so basic that it's hard to know what to assume (although if one goes through the trouble of carefully axiomatizing math with something like set theory, then it's possible to give a formal proof using a technique called mathematical induction that we will discuss in Chapter 2). At any rate, the key fact is that since A and B are disjoint, they have no elements in common. Therefore, each element of $A \cup B$ is in exactly one of A or B. Notice that the assumption that A and B are disjoint is essential. If $A = \{1, 2\}$ and $B = \{2, 3\}$, then $|A| = 2 = |B|$, but $|A \cup B| = 3$ because $A \cup B = \{1, 2, 3\}$.

Although the next result is again very intuitive, we show how to prove it using the Sum Rule.

Proposition 1.2.4 (Complement Rule)**.** *If A and B are finite sets and $B \subseteq A$, then $|A \backslash B| = |A| - |B|$.*

Proof. Notice that $A \backslash B$ and B are disjoint sets and that $(A \backslash B) \cup B = A$. Using the Sum Rule, it follows that $|A \backslash B| + |B| = |A|$. Subtracting $|B|$ from both sides, we conclude that $|A \backslash B| = |A| - |B|$. □

We can now easily generalize this to the case where B might not be a subset of A.

Proposition 1.2.5 (General Complement Rule)**.** *If A and B are finite sets, then $|A \backslash B| = |A| - |A \cap B|$.*

Proof. We have $A \backslash B = A \backslash (A \cap B)$. Since $A \cap B \subseteq A$, we can now apply the Complement Rule. □

We can generalize the Sum Rule to the following.

Definition 1.2.6. *A collection of sets $A_1, A_2, \ldots, A_n$ is* pairwise disjoint *if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.*

Fact 1.2.7 (General Sum Rule)**.** *If $A_1, A_2, \ldots, A_n$ are finite sets that are pairwise disjoint, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.*

Again, we won't give a formal proof of this fact (although it it possible to do so from the Sum Rule by induction on n). Notice that the pairwise disjoint assumption is again key, and it's not even enough to assume that $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$ (see the homework).

Proposition 1.2.8. *If A and B are finite sets, we have $|A \cup B| = |A| + |B| - |A \cap B|$.*

Proof. Consider the three sets $A\backslash B$, $B\backslash A$, and $A\cap B$. These three sets are pairwise disjoint, and their union is $A\cup B$. Using the General Sum Rule, we conclude that

$$|A\cup B| = |A\backslash B| + |B\backslash A| + |A\cap B|.$$

Now $|A\backslash B| = |A| - |A\cap B|$ and $|B\backslash A| = |B| - |A\cap B|$ by the General Complement Rule. Plugging these in, we conclude that

$$|A\cup B| = |A| - |A\cap B| + |B| - |A\cap B| + |A\cap B|,$$

and hence

$$|A\cup B| = |A| + |B| - |A\cap B|.$$

□

Proposition 1.2.9 (Product Rule)**.** *If A and B are finite sets, then $|A\times B| = |A|\cdot|B|$.*

Proof. Let $n = |A|$ and let $m = |B|$. List the elements of A so that $A = \{a_1, a_2, \ldots, a_n\}$. Similarly, list the elements of B so that $B = \{b_1, b_2, \ldots, b_m\}$. For each i, let

$$A_i = \{(a_i, b_j) : 1 \le j \le m\} = \{(a_i, b_1), (a_i, b_2), \ldots, (a_i, b_m)\}.$$

Thus, A_i is the subset of $A\times B$ consisting only of those pairs whose first element is a_i. Notice that the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint and that

$$A\times B = A_1\cup A_2\cup\cdots\cup A_n.$$

Furthermore, we have that $|A_i| = m$ for all i. Using the General Sum Rule, we conclude that

$$\begin{aligned} |A\times B| &= |A_1| + |A_2| + \cdots + |A_n| \\ &= m + m + \cdots + m \\ &= n\cdot m \\ &= |A|\cdot|B|. \end{aligned}$$

□

Using induction (again, see Chapter 2), one can prove the following generalization.

Proposition 1.2.10 (General Product Rule)**.** *If $A_1, A_2, \ldots, A_n$ are finite sets, then $|A_1\times A_2\times\cdots\times A_n| = |A_1|\cdot|A_2|\cdots|A_n|$.*

Corollary 1.2.11. *If A is a finite set and $n\in\mathbb{N}^+$, then $|A^n| = |A|^n$.*

Corollary 1.2.12. *For any $n\in\mathbb{N}^+$, we have that $|\{0,1\}^n| = 2^n$, i.e. there are 2^n many sequences of 0's and 1's of length n.*

1.3 Relations and Equivalence Relations

Definition 1.3.1. *Let A and B be sets. A* (binary) relation *between A and B is a subset $R\subseteq A\times B$. If $A = B$, then we call a subset of $A\times A$ a* (binary) relation on A.

For example, let $A = \{1, 2, 3\}$ and $B = \{6, 8\}$ as above. We saw above that

$$\{1, 2, 3\} \times \{6, 8\} = \{(1, 6), (1, 8), (2, 6), (2, 8), (3, 6), (3, 8)\}.$$

The set

$$R = \{(1, 6), (1, 8), (3, 8)\}$$

is a relation between A and B, although certainly not a very interesting one. However, we'll use it to illustrate a few facts. First, in a relation, it's possible for an element of A to be related to multiple elements of B, as in the case for $1 \in A$ for our example R. Also, it's possible that an element of A is related to no elements of B, as in the case of $2 \in A$ for our example R.

For a more interesting example, consider the binary relation on $\mathbb{Z}$ defined by $R = \{(a, b) \in \mathbb{Z}^2 : a < b\}$. Notice that $(4, 7) \in R$ and $(5, 5) \notin R$.

By definition, relations are sets. However, it is typically cumbersome to use set notation to write things like $(1, 6) \in R$. Instead, it usually makes much more sense to use infix notation and write $1R6$. Moreover, we can use better notation for the relation by using a symbol like $\sim$ instead of R. In this case, we would write $1 \sim 6$ instead of $(1, 6) \in \sim$ or $2 \not\sim 8$ instead of $(2, 8) \notin \sim$.

With this new notation, we give a few examples of binary relations on $\mathbb{R}$:

- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $x^2 + y^2 = 1$.
- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $x^2 + y^2 \leq 1$.
- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $x = \sin y$.
- Given $x, y \in \mathbb{R}$, we let $x \sim y$ if $y = \sin x$.

Again, notice from these examples that given $x \in \mathbb{R}$, there might be 0, 1, 2, or even infinitely many $y \in \mathbb{R}$ with $x \sim y$.

If we let $A = \{0, 1\}^*$ be the set of all finite sequences of 0's and 1's, then the following are binary relations on A:

- Given $\sigma, \tau \in A$, we let $\sigma \sim \tau$ if σ and τ have the same number of 1's.
- Given $\sigma, \tau \in A$, we let $\sigma \sim \tau$ if σ occurs as a consecutive subsequence of τ (for example, we have $010 \sim 001101011$ because 010 appears in positions 5-6-7 of 001101011).

For a final example, let A be the set consisting of the 50 states. Let R be the subset of $A \times A$ consisting of those pairs of states that have a common letter in the second position of their postal codes. For example, we have (Iowa,California) $\in R$ and and (Iowa, Virginia) $\in R$ because the postal codes of these sets are IA, CA, VA. We also have (Minnesota, Tennessee) $\in R$ because the corresponding postal codes are MN and TN. Now (Texas, Texas) $\in R$, but there is no $a \in A$ with $a \neq$ Texas such that (Texas, a) $\in R$, because no other state has X as the second letter of its postal code. Texas stands alone.

Recall that a binary relation on a set A is *any* subset of $A \times A$. As a result, a given relation might have very few nice properties. However, there are many special classes of relations, and one of the most important types is the following.

Definition 1.3.2. *An* equivalence relation *on a set A is a binary relation $\sim$ on A having the following three properties:*

- *$\sim$ is reflexive: $a \sim a$ for all $a \in A$.*
- *$\sim$ is symmetric: Whenever $a, b \in A$ satisfy $a \sim b$, we have $b \sim a$.*
- *$\sim$ is transitive: Whenever $a, b, c \in A$ satisfy $a \sim b$ and $b \sim c$, we have $a \sim c$.*

Consider the binary relation $\sim$ on $\mathbb{Z}$ where $a \sim b$ means that $a \leq b$. Notice that $\sim$ is reflexive because $a \leq a$ for all $a \in \mathbb{Z}$. Also, $\sim$ is transitive because if $a \leq b$ and $b \leq c$, then $a \leq c$. However, $\sim$ is not symmetric because $3 \sim 4$ but $4 \not\sim 3$. Thus, although $\sim$ satisfies two out of the three requirements, it is not an equivalence relation.

A simple example of an equivalence relation is where $A = \mathbb{R}$ and $a \sim b$ means that $|a| = |b|$. In this case, it is straightforward to check that $\sim$ is an equivalence relation. We now move on to some more interesting examples which we treat more carefully.

Example 1.3.3. *Let A be the set of all $n \times n$ matrices with real entries. Let $M \sim N$ mean that there exists an invertible $n \times n$ matrix P such that $M = PNP^{-1}$. We then have that $\sim$ is an equivalence relation on A.*

Proof. We need to check the three properties.

- Reflexive: Let $M \in A$ be arbitrary. The $n \times n$ identity matrix I is invertible and satisfies $I^{-1} = I$, so we have $M = IMI^{-1}$. Therefore, $\sim$ is reflexive.

- Symmetric: Let $M, N \in A$ be arbitrary with $M \sim N$. Fix an $n \times n$ invertible matrix P with $M = PNP^{-1}$. Multiplying on the left by P^{-1} we get $P^{-1}M = NP^{-1}$, and now multiplying on the right by P we conclude that $P^{-1}MP = N$. We know from linear algebra that P^{-1} is also invertible and $(P^{-1})^{-1} = P$, so $N = P^{-1}M(P^{-1})^{-1}$ and hence $N \sim M$.

- Transitive: Let $L, M, N \in A$ be arbitrary with $L \sim M$ and $M \sim N$. Since $L \sim M$, we may fix an $n \times n$ invertible matrix P with $L = PMP^{-1}$. Since $M \sim N$, we may fix an $n \times n$ invertible matrix Q with $M = QNQ^{-1}$. We then have

  $$L = PMP^{-1} = P(QNQ^{-1})P^{-1} = (PQ)N(Q^{-1}P^{-1}).$$

  Now by linear algebra, we know that the product of two invertible matrices is invertible, so PQ is invertible and furthermore we know that $(PQ)^{-1} = Q^{-1}P^{-1}$. Therefore, we have

  $$L = (PQ)N(PQ)^{-1},$$

  so $L \sim N$.

Putting it all together, we conclude that $\sim$ is an equivalence relation on A. □

Example 1.3.4. *Let A be the set $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$, i.e. A is the set of all pairs $(a, b) \in \mathbb{Z}^2$ with $b \neq 0$. Define a relation $\sim$ on A as follows. Given $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$, we let $(a, b) \sim (c, d)$ mean $ad = bc$. We then have that $\sim$ is an equivalence relation on A.*

Proof. We check the three properties.

- Reflexive: Let $a, b \in \mathbb{Z}$ be arbitrary with $b \neq 0$. Since $ab = ba$, it follows that $(a, b) \sim (a, b)$.

- Symmetric: Let $a, b, c, d \in \mathbb{Z}$ be arbitrary with $b, d \neq 0$, and $(a, b) \sim (c, d)$. We then have that $ad = bc$. From this, we conclude that $cb = da$ so $(c, d) \sim (a, b)$.

- Transitive: Let $a, b, c, d, e, f \in \mathbb{Z}$ be arbitrary with $b, d, f \neq 0$ where $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. We then have that $ad = bc$ and $cf = de$. Multiplying the first equation by f we see that $adf = bcf$. Multiplying the second equation by b gives $bcf = bde$. Therefore, we know that $adf = bde$. Now $d \neq 0$ by assumption, so we may cancel it to conclude that $af = be$. It follows that $(a, b) \sim (e, f)$

Therefore, $\sim$ is an equivalence relation on A. □

Let's analyze the above situation more carefully. We have $(1,2) \sim (2,4)$, $(1,2) \sim (4,8)$, $(1,2) \sim (-5,-10)$, etc. If we think of (a,b) as representing the fraction $\frac{a}{b}$, then the relation $(a,b) \sim (c,d)$ is saying exactly that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal. You may never have thought about equality of fractions as the result of imposing an equivalence relation on pairs of integers, but that is exactly what it is. We will be more precise about this below.

Definition 1.3.5. *Let $\sim$ be an equivalence relation on a set A. Given $a \in A$, we let*

$$\overline{a} = \{b \in A : a \sim b\}.$$

The set $\overline{a}$ is called the equivalence class *of a.*

Some sources use the notation $[a]$ instead of $\overline{a}$. The former notation helps emphasize that the equivalence class of a is a *subset* of A rather than an element of A. However, it is cumbersome notation to use when we begin working with equivalence classes. We will stick with our notation, although it might take a little time to get used to it. Notice that by the reflexive property of $\sim$, we have that $a \in \overline{a}$ for all $a \in A$.

For example, let's return to where A is the set consisting of the 50 states and R is the subset of $A \times A$ consisting of those pairs of states that have a common letter in the second position of their postal codes. It's straightforward to show that R is an equivalence relation on A. We have

$$\overline{\text{Iowa}} = \{\text{California}, \text{Georgia}, \text{Iowa}, \text{Louisiana}, \text{Massachusetts}, \text{Pennsylvania}, \text{Virginia}, \text{Washington}\},$$

while

$$\overline{\text{Minnesota}} = \{\text{Indiana}, \text{Minnesota}, \text{Tennessee}\}$$

and

$$\overline{\text{Texas}} = \{\text{Texas}\}.$$

Notice that each of these are sets, even in the case of $\overline{\text{Texas}}$.

For another example, suppose we are working with $A = \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ where $(a,b) \sim (c,d)$ means that $ad = bc$. As discussed above, some elements of $\overline{(1,2)}$ are $(1,2)$, $(2,4)$, $(4,8)$, $(-5,-10)$, etc. So

$$\overline{(1,2)} = \{(1,2), (2,4), (4,8), (-5,-10), \dots\}.$$

Again, I want to emphasize that $\overline{(a,b)}$ is a subset of A.

The following proposition is hugely fundamental. It says that if two equivalence classes overlap, then they must in fact be equal. In other words, if $\sim$ is an equivalence on A, then the equivalence classes *partition* the set A into pieces.

Proposition 1.3.6. *Let $\sim$ be an equivalence relation on a set A and let $a, b \in A$. If $\overline{a} \cap \overline{b} \neq \emptyset$, then $\overline{a} = \overline{b}$.*

Proof. Suppose that $\overline{a} \cap \overline{b} \neq \emptyset$. Since this set is nonempty, we can fix some $c \in \overline{a} \cap \overline{b}$. We then have $a \sim c$ and $b \sim c$. By symmetry, we know that $c \sim b$, and using transitivity we get that $a \sim b$. Using symmetry again, we conclude that $b \sim a$. We now show that $\overline{a} = \overline{b}$ by showing each containment:

- We first show that $\overline{a} \subseteq \overline{b}$. Let $x \in \overline{a}$ be arbitrary. We then have that $a \sim x$. Since $b \sim a$ from above, we can use transitivity to conclude that $b \sim x$, hence $x \in \overline{b}$.
- We next show that $\overline{b} \subseteq \overline{a}$. Let $x \in \overline{b}$ be arbitrary. We then have that $b \sim x$. Since $a \sim b$ from above, we can use transitivity to conclude that $a \sim x$, hence $x \in \overline{a}$.

Putting this together, we conclude that $\overline{a} = \overline{b}$. □

With that proposition in hand, we are ready for the foundational theorem about equivalence relations.

Theorem 1.3.7. *Let $\sim$ be an equivalence relation on a set A and let $a, b \in A$.*

1. *$a \sim b$ if and only if $\overline{a} = \overline{b}$.*
2. *$a \not\sim b$ if and only if $\overline{a} \cap \overline{b} = \emptyset$.*

Proof. 1. Suppose first that $a \sim b$. We then have that $b \in \overline{a}$. Now we know that $b \sim b$ because $\sim$ is reflexive, so $b \in \overline{b}$. Thus, $b \in \overline{a} \cap \overline{b}$, so $\overline{a} \cap \overline{b} \neq \emptyset$. Using Proposition 1.3.6, we conclude that $\overline{a} = \overline{b}$.

Suppose conversely that $\overline{a} = \overline{b}$. Since $b \sim b$ because $\sim$ is reflexive, we have that $b \in \overline{b}$. Therefore, $b \in \overline{a}$ and hence $a \sim b$.

2. Suppose that $a \not\sim b$. Since we just proved (1), it follows that $\overline{a} \neq \overline{b}$, so by Proposition 1.3.6 we must have $\overline{a} \cap \overline{b} = \emptyset$.

Suppose conversely that $\overline{a} \cap \overline{b} = \emptyset$. We then have $\overline{a} \neq \overline{b}$ (because $a \in \overline{a}$ so $\overline{a} \neq \emptyset$), so $a \not\sim b$ by part (1). □

Therefore, given an equivalence relation $\sim$ on a set A, the equivalence classes partition A into pieces. Working out the details in our postal code example, one can show that $\sim$ has 1 equivalence class of size 8 (namely $\overline{\text{Iowa}}$, which is the same set as $\overline{\text{California}}$ and 6 others), 3 equivalence classes of size 4, 4 equivalence classes of size 3, 7 equivalence classes of size 2, and 4 equivalence classes of size 1.

Let's revisit the example of $A = \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ where $(a, b) \sim (c, d)$ means $ad = bc$. The equivalence class of $(1, 2)$, namely the set $\overline{(1, 2)}$ is the set of all pairs of integers which are ways of representing the fraction $\frac{1}{2}$. In fact, this is how once can "construct" the rational numbers from the integers. We simply *define* the rational numbers to be the set of equivalence classes of A under $\sim$. In other words, we let

$$\frac{a}{b} = \overline{(a, b)}.$$

So when we write something like

$$\frac{1}{2} = \frac{4}{8},$$

we are simply saying that

$$\overline{(1, 2)} = \overline{(4, 8)},$$

which is true because $(1, 2) \sim (4, 8)$.

1.4 Functions

We're all familiar with functions from Calculus. In that context, a function is often given by a "formula", such as $f(x) = x^4 - 4x^3 + 2x - 1$. However, we also encounter piecewise-defined functions, such as

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 2, \\ x - 1 & \text{if } x < 2, \end{cases}$$

and the function $g(x) = |x|$, which is really piecewise defined as

$$g(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

For a more exotic example of a piecewise defined function, consider

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Despite these examples, only the most basic functions in mathematics are defined through formulas on pieces. For instance, the function $f(x) = \sin x$ is *not* given by a formula, and it is difficult to compute values of this function with any accuracy using only basic operations like $+$ and $\cdot$. In fact, we give this function the strange new name of "sine" because we can not express it easily using more basic operations. The function $f(x) = 2^x$ is easy to compute for $x \in \mathbb{Z}$, but it's actually nontrivial to compute and even define in general (after all, do you remember the definition of 2^π?). Even more fundamentally, the function $f(x) = \sqrt{x}$ is also not really given by a formula, because the definition, i.e. $f(x)$ is the unique positive y with the property that $y^2 = x$, does not give us an easy way to compute it.

Beyond these fundamental functions that you encounter before Calculus, you learn more exotic ways to define functions in Calculus. Given a function f, you learn how to define a new function f', called the derivative of f, using a certain limit operation. Now in many cases, you can compute f' more easily using facts like the Product Rule and the Chain Rule, but these rules do not always apply. Moreover, given any continuous function g, we can define a new function f by letting

$$f(x) = \int_0^x g(t)\ dt.$$

In other words, f is defined as the "(signed) area of g so far" function, in that $f(x)$ is defined to be the (signed) area between the graph of g and the x-axis over the interval from 0 to x. Formally, f is defined as a limit of Riemann sums. Again, in Calculus you learn ways to compute $f(x)$ more easily in many special cases using the Fundamental Theorem of Calculus. For example, if

$$f(x) = \int_0^x (3t^2 + t)\ dt,$$

then we can also compute f as

$$f(x) = x^3 + \frac{x^2}{2},$$

while if

$$f(x) = \int_0^x \sin t\ dt,$$

then we can also compute f as

$$f(x) = 1 - \cos x.$$

However, not all integrals can be evaluated so easily. In fact, it turns out that the perfectly well-defined function

$$f(x) = \int_0^x e^{-t^2}\ dt$$

can not be expressed through polynomials, exponentials, logs, and trigonometric functions using only operations like $+$, $\cdot$, and function composition. Of course, we can still approximate it using Riemann sums (or Simpson's Rule), and this is important for us to be able to do since this function represents the area under a normal curve, which is essential in statistics.

If we move away from functions whose inputs and outputs are real numbers, we can think about other interesting ways to define functions. For example, suppose we define a function whose inputs and outputs are elements of $\mathbb{R}^2$ by letting $f(\vec{u})$ be the result of rotating $\vec{u}$ by 27^o clockwise around the origin. This seems to be a well-defined function despite the fact that it is not clear how to compute it (though you likely learned how to compute it in Linear Algebra).

Alternatively, consider a function whose inputs and outputs are natural numbers by letting $f(n)$ be the number of primes less than or equal to n. For example, we have $f(3) = 2$, $f(4) = 2$, $f(9) = 4$, and $f(30) = 10$. Although it is possible to compute this function, it's not clear whether we can compute it quickly. In other words, it's not obvious if we can compute something like $f(2^{50})$ without a huge amount of work.

Perhaps you have some exposure to the concept of a function as it is used in computer programming. From this perspective, a function is determined by a sequence of imperative statements or function compositions as defined by a precise programming language. Since a computer is doing the interpreting, of course all such functions can be computed in principle (or if your computations involve real numbers, then at least up to good approximations). However, if you take this perspective, an interesting question arises. If we write two different functions f and g that do not follow the same steps, and perhaps even act qualitatively differently in structure, but they always produce the same output on the same input, should we consider them to be the same function? We can even ask this question outside of the computer science paradigm. For example, if we define $f(x) = \sin^2 x + \cos^2 x$ and $g(x) = 1$, then should we consider f and g be the same function?

We need to make a choice about how to define a function in general. Intuitively, given two sets A and B, a function $f\colon A \to B$ is an input-output "mechanism" that produces a *unique* output $b \in B$ for any given input $a \in A$. As we've seen, the vast majority of functions that we have encountered so far can be computed in principle, so up until this point, we could interpret "mechanism" in an algorithmic and computational sense. However, we want to allow as much freedom as possible in this definition so that we can consider new ways to define functions in time. In fact, as you might see in later courses (like Automata, Formal Languages, and Computational Complexity), there are some natural functions that are not computable even in theory. As a result, we choose to abandon the notion of computation in our definition. By making this choice, we will be able to sidestep some of the issues in the previous paragraph, but we still need to make a choice about whether to consider the functions $f(x) = \sin^2 x + \cos^2 x$ and $g(x) = 1$ to be equal.

With all of this background, we are now in a position to define functions as certain special types of sets. Thinking about functions from this more abstract point of view eliminates the vague "mechanism" concept because they will simply be sets. With this perspective, we'll see that functions can be defined in any way that a set can be defined. Our approach both clarifies the concept of a function and also provides us with some much needed flexibility in defining functions in more interesting ways. Here is the formal definition.

Definition 1.4.1. *Let A and B be sets. A* function *from A to B is a subset f of $A \times B$ with the property that for all $a \in A$, there exists a unique $b \in B$ with $(a, b) \in f$. Also, instead of writing "f is a function from A to B", we typically use the shorthand notation "$f\colon A \to B$".*

For example, let $A = \{2, 3, 5, 7\}$ and let $B = \mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$. An example of a function $f\colon A \to B$ is the set

$$f = \{(2, 71), (3, 4), (5, 9382), (7, 4)\}.$$

Notice that in the definition of a function from A to B, we know that for every $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$. However, as this example shows, it may not be the case that for every $b \in B$, there is a unique $a \in A$ with $(a, b) \in f$. Be careful with the order of quantifiers!

We can also convert the typical way of defining a function into this formal set theoretic way. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ by letting $f(x) = x^2$. We can instead define f by the set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\},$$

or parametrically as

$$\{(x, x^2) : x \in \mathbb{R}\}.$$

One side effect of our definition of a function is that we immediately obtain a nice definition for when two functions $f\colon A \to B$ and $g\colon A \to B$ are equal because we have defined when two sets are equal. Given two function $f\colon A \to B$ and $g\colon A \to B$, if we unwrap our definition of set equality, we see that $f = g$ exactly when f and g have the same elements, which is precisely the same thing as saying that $f(a) = g(a)$ for all $a \in A$. In particular, the *manner* in which we describe functions does not matter so long as the functions behave the same on all inputs. For example, if we define $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ by letting $f(x) = \sin^2 x + \cos^2 x$ and $g(x) = 1$, then we have that $f = g$ because $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Thinking of functions as special types of sets is helpful to clarify definitions, but is often awkward to work with in practice. For example, writing $(2, 71) \in f$ to mean that f sends 2 to 71 quickly becomes annoying. Thus, we introduce some new notation matching up with our old experience with functions.

Notation 1.4.2. *Let A and B be sets. If $f\colon A \to B$ and $a \in A$, we write $f(a)$ to mean the unique $b \in B$ such that $(a, b) \in f$.*

For instance, in the above example of f, we can instead write

$$f(2) = 71, \qquad f(3) = 4, \qquad f(5) = 9382, \qquad \text{and} \qquad f(7) = 4.$$

Definition 1.4.3. *Let $f\colon A \to B$ be a function.*

- *We call A the* domain *of f.*
- *We call B the* codomain *of f.*
- *We define range$(f) = \{b \in B : \text{There exists } a \in A \text{ with } f(a) = b\}$.*

Notice that given a function $f\colon A \to B$, we have range$(f) \subseteq B$, but it is possible that range$(f) \neq B$. For example, in the above case, we have that the codomain of f is $\mathbb{N}$, but range$(f) = \{4, 71, 9382\}$. In general, given a function $f\colon A \to B$, it may be very difficult to determine range(f) because we may need to search through all $a \in A$.

For an interesting example of a function with a mysterious looking range, fix $n \in \mathbb{N}^+$ and define $f\colon \{0, 1, 2, \ldots, n-1\} \to \{0, 1, 2, \ldots, n-1\}$ by letting $f(a)$ be the remainder when dividing a^2 by n. For example, if $n = 10$, then we have the following table of values of f:

$$\begin{array}{lllll} f(0) = 0 & f(1) = 1 & f(2) = 4 & f(3) = 9 & f(4) = 6 \\ f(5) = 5 & f(6) = 6 & f(7) = 9 & f(8) = 4 & f(9) = 1. \end{array}$$

Thus, for $n = 10$, we have range$(f) = \{0, 1, 4, 5, 6, 9\}$. This simple but strange looking function has many interesting properties. Given a reasonably large number $n \in \mathbb{N}$, it looks potentially difficult to determine whether an element is in range(f) because we might need to search through a huge number of inputs to see if a given output actually occurs. If n is prime, then it turns out that there are much faster ways to determine if a given element is in range(f) (see Algebraic Number Theory). However, it is widely believed (although we do not currently have a proof!) that there is no efficient method to do this when n is the product of two large primes, and this is the basis for some cryptosystems (Goldwasser-Micali) and pseudo-random number generators (Blum-Blum-Shub).

Definition 1.4.4. *Suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. The* composition *of g and f, denoted $g \circ f$, is the function $g \circ f\colon A \to C$ defined by letting $(g \circ f)(a) = g(f(a))$ for all $a \in A$.*

Notice that in general we have $f \circ g \neq g \circ f$ even when both are defined! If $f\colon \mathbb{R} \to \mathbb{R}$ is $f(x) = x + 1$ and $g\colon \mathbb{R} \to \mathbb{R}$ is $g(x) = x^2$, then

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(x^2) \\ &= x^2 + 1 \end{aligned}$$

while

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(x + 1) \\ &= (x + 1)^2 \\ &= x^2 + 2x + 1. \end{aligned}$$

Notice then that $(f \circ g)(1) = 1^2 + 1 = 2$ while $(g \circ f)(1) = 1^2 + 2 \cdot 1 + 1 = 4$. Since we have found one example of an $x \in \mathbb{R}$ with $(f \circ g)(x) \neq (f \circ g)(x)$, we conclude that $f \circ g \neq g \circ f$. It does not matter that there do exist some values of x with $(f \circ g)(x) = (f \circ g)(x)$ (for example, this is true when $x = 0$). Remember that two functions are equal precisely when they agree on *all* inputs, so to show that the two functions are not equal it suffices to find just one value where they disagree (again remember that the negation of a "for all" statement is a "there exists" statement).

Proposition 1.4.5. *Let A, B, C, D be sets. Suppose that $f\colon A \to B$, that $g\colon B \to C$, and that $h\colon C \to D$ are functions. We then have that $(h \circ g) \circ f = h \circ (g \circ f)$. Stated more simply, function composition is associative whenever it is defined.*

Proof. Let $a \in A$ be arbitrary. We then have

$$\begin{aligned}((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) \\ &= h(g(f(a))) \\ &= h((g \circ f)(a)) \\ &= (h \circ (g \circ f))(a),\end{aligned}$$

where each step follows by definition of composition. Therefore $((h \circ g) \circ f)(a) = (h \circ (g \circ f))(a)$ for all $a \in A$. It follows that $(h \circ g) \circ f = h \circ (g \circ f)$. □

1.5 Divisibility

Definition 1.5.1. *Let $a, b \in \mathbb{Z}$. We say that a* divides *b, and write $a \mid b$, if there exists $m \in \mathbb{Z}$ with $b = am$.*

For example, we have $2 \mid 6$ because $2 \cdot 3 = 6$ and $3 \mid -21$ because $3 \cdot (-7) = 21$. On the other hand, we have $2 \nmid 5$. To see this, we argue as follows.

- For any $m \in \mathbb{Z}$ with $m \leq 2$, we have $2m \leq 4$.
- For any $m \in \mathbb{Z}$ with $m > 2$, we have $m \geq 3$, so $2m \geq 6$.

Therefore, for every $m \in \mathbb{Z}$, we have $2m \neq 5$. It follows that $2 \nmid 5$. We will see less painful ways to prove this later.

Notice that $a \mid 0$ for every $a \in \mathbb{Z}$ because $a \cdot 0 = 0$ for all $a \in \mathbb{Z}$. In particular, we have $0 \mid 0$ because as noted we have $0 \cdot 0 = 0$. Of course we also have $0 \cdot 3 = 0$ and in fact $0 \cdot m = 0$ for all $m \in \mathbb{Z}$, so every integer serves as a "witness" that $0 \mid 0$. Our definition says nothing about the $m \in \mathbb{Z}$ being unique.

For example, we have $2 \mid 6$ because $2 \cdot 3 = 6$ and $-3 \mid 21$ because $-3 \cdot 7 = 21$. We also have that $2 \nmid 5$ since it is "obvious" that no such integer exists. If you are uncomfortable with that (and you should be!), we will give methods to prove such statements in the next couple of sections.

Proposition 1.5.2. *If $a \mid b$ and $b \mid c$, then $a \mid c$.*

Proof. Since $a \mid b$, there exists $m \in \mathbb{Z}$ with $b = am$. Since $b \mid c$, there exists $n \in \mathbb{Z}$ with $c = bn$. We then have

$$c = bn = (am)n = a(mn)$$

Since $mn \in \mathbb{Z}$, it follows that $a \mid c$. □

Proposition 1.5.3.

1. *If $a \mid b$, then $a \mid kb$ for all $k \in \mathbb{Z}$.*

2. *If $a \mid b$ and $a \mid c$, then $a \mid (b+c)$.*

3. *If $a \mid b$ and $a \mid c$, then $a \mid (mb+nc)$ for all $m, n \in \mathbb{Z}$.*

Proof.

1. Let $a, b, k \in \mathbb{Z}$ be arbitrary with $a \mid b$. Since $a \mid b$, we can fix $m \in \mathbb{Z}$ with $b = am$. We then have
$$kb = k(am) = a(mk).$$
Since $mk \in \mathbb{Z}$, it follows that $a \mid kb$.

2. Let $a, b, c \in \mathbb{Z}$ be arbitrary with both $a \mid b$ and $a \mid c$. Since $a \mid b$, we can fix $m \in \mathbb{Z}$ with $b = am$. Since $a \mid c$, we can fix $n \in \mathbb{Z}$ with $c = an$. Notice that
$$b + c = am + an = a(m + n).$$
Since $m, n \in \mathbb{Z}$, we know that $m + n \in \mathbb{Z}$, it follows that $a \mid b + c$.

3. Let $m, n \in \mathbb{Z}$ be arbitrary. Since $a \mid b$, we conclude from part 1 that $a \mid mb$. Since $a \mid c$, we conclude from part 1 again that $a \mid nc$. Using part 2, it follows that $a \mid (bm + cn)$.

□

Proposition 1.5.4. *Suppose that $a, b \in \mathbb{Z}$. If $a \mid b$ and $b \neq 0$, then $|a| \leq |b|$.*

Proof. Suppose that $a \mid b$ with $b \neq 0$. Fix $d \in \mathbb{Z}$ with $ad = b$. Since $b \neq 0$, we have $d \neq 0$. Thus, $|d| \geq 1$, and so
$$\begin{aligned} |b| &= |ad| \\ &= |a| \cdot |d| \\ &\geq |a| \cdot 1 \\ &= |a|. \end{aligned}$$

□

Corollary 1.5.5. *Suppose that $a, b \in \mathbb{Z}$. If $a \mid b$ and $b \mid a$, then either $a = b$ or $a = -b$.*

Proof. We handle three cases:

- *Case 1:* Suppose that $a \neq 0$ and $b \neq 0$. By the Proposition 1.5.4, we know that both $|a| \leq |b|$ and $|b| \leq |a|$. It follows that $|a| = |b|$, and hence either $a = b$ or $a = -b$.
- *Case 2:* Suppose that $a = 0$. Since $a \mid b$, we may fix $m \in \mathbb{Z}$ with $b = am$. We then have $b = am = 0m = 0$ as well. Therefore, $a = b$.
- *Case 3:* Suppose that $b = 0$. Since $b \mid a$, we may fix $m \in \mathbb{Z}$ with $a = bm$. We then have $a = bm = 0m = 0$ as well. Therefore, $a = b$.

Thus, in all cases, we have that either $a = b$ or $a = -b$. □

Given an integer $a \in \mathbb{Z}$, we introduce the following notation for the set of all divisors of a.

Definition 1.5.6. *Given $a \in \mathbb{Z}$, we let $Div(a) = \{d \in \mathbb{Z} : d \mid a\}$ and we let $Div^+(a) = \{d \in \mathbb{N}^+ : d \mid a\}$.*

Notice that if $a \neq 0$, then $|d| \leq |a|$ for all $d \in Div(a)$ by Proposition 1.5.4. Thus, we need only check finitely many values to determine $Div(a)$. For instance, we have $Div(7) = \{1, -1, 7, -7\}$ (don't forget the negatives!), which we can write more succinctly as $\{\pm 1, \pm 7\}$, while $Div(6) = \{\pm 1, \pm 2, \pm 3, \pm 6\}$ (we'll soon see better ways to compute these sets that do not require an exhaustive search). For a more interesting example, we have $Div(0) = \mathbb{Z}$.

Proposition 1.5.7. *For any $a \in \mathbb{Z}$, we have $Div(a) = Div(-a)$.*

Proof. Exercise. □

Chapter 2

Induction and Well-Ordering

2.1 Mathematical Induction

Suppose that we want to prove that a certain statement is true for all natural numbers. In other words, we want to do the following:

- Prove that the statement is true for 0.
- Prove that the statement is true for 1.
- Prove that the statement is true for 2.
- Prove that the statement is true for 3.
- ....

Of course, since there are infinitely many natural numbers, going through each one in turn does not work because we will never handle them all this way. How can we get around this? Suppose that when we examine the first few proofs above that they look the same except that we replace 0 by 1 everywhere, or 0 by 2 everywhere, etc. In this case, one is tempted to say that "the pattern continues" or something similar, but that is not convincing because we can't be sure that the pattern does not break down when we reach 5419. One way to argue that the "the pattern continues" and handle all of the infinitely many possibilities at once is to take an arbitrary natural number n, and prove that the statement is true for n using *only* the fact that n is a natural number (but *not* any particular natural number).

The method of taking an arbitrary $n \in \mathbb{N}$ and proving that the statement is true for n is the standard way of proving a statement involving a "for all" quantifier. This technique also works to prove that a statement is true for all real numbers or for all matrices, as long as we take an *arbitrary* such object. However, there is a different method one can use to prove that every natural number has a certain property, and this one does not carry over to other settings like the real numbers. The key fact is that the natural numbers start with 0 and proceed in discrete steps forward. With this in mind, consider what would happen if we could accomplish each of the following:

- Prove that the statement is true for 0.
- Prove that if the statement is true for 0, then the statement is true for 1.
- Prove that if the statement is true for 1, then the statement is true for 2.
- Prove that if the statement is true for 2, then the statement is true for 3.

- ....

Suppose that we are successful in proving each of these. From the first line, we then know that the statement is true for 0. Since we now know that it's true for 0, we can use the second line to conclude that the statement is true for 1. Since we now know that it's true for 1, we can use the second line to conclude that the statement is true for 2. And so on. In the end, we are able to conclude that the statement is true for all natural numbers.

Let's examine this situation more closely. On the fact of it, each line looks more complicated than the corresponding line for for a direct proof. However, the key fact is that from the second line onward, we now have an additional assumption! Thus, instead of proving that the statement is true for 3 without any help, we can now use the assumption that the statement is true for 2 in that argument. Extra assumptions are always welcome because we have more that we can use in the actual argument.

Of course, as in our discussion at the beginning of this section, we can't hope to prove each of these infinitely many things one at a time. In an ideal world, the arguments from the second line onward all look exactly the same with the exception of replacing the number involved. Thus, the idea is to prove the following:

- Prove that the statement is true for 0.
- Prove that if the statement is true for n, then the statement is true for $n + 1$.

Notice that for the second line, we would need to prove that it is true for an arbitrary $n \in \mathbb{N}$, just like we would have to in a direct argument. An argument using these method is called a proof by (mathematical) *induction*, and it is an extremely useful and common technique in mathematics. We can also state this approach formally in terms of sets, allowing us to bypass the vague notion of "statement" that we employed above.

Fact 2.1.1 (Principle of Mathematical Induction on $\mathbb{N}$)**.** *Let $X \subseteq \mathbb{N}$. Suppose that the following are true:*

- $0 \in X$ *(the* base case*).*
- $n + 1 \in X$ *whenever* $n \in X$ *(the* inductive step*).*

We then have that $X = \mathbb{N}$.

Once again, here's the intuitive argument for why induction is valid. By the first assumption, we know that $0 \in X$. Since $0 \in X$, the second assumption tells us that $1 \in X$. Since $1 \in X$, the second assumption again tells us that $2 \in X$. By repeatedly applying the second assumption in this manner, each element of $\mathbb{N}$ is eventually determined to be in X. Notice that a similar argument works if we start with a different base case, i.e. if we start by proving that $3 \in X$ and then prove the inductive step, then it follows that $n \in X$ for all $n \in \mathbb{N}$ with $n \geq 3$.

Although we have stated induction with a base case of 0, it is also possible to give an inductive proof that starts at a different natural. For example, if we prove a base case the $4 \in X$, and we prove the usual inductive step that $n + 1 \in X$ whenever $n \in X$, then we can conclude that $n \in X$ for all $n \in \mathbb{N}$ with $n \geq 4$, i.e. that $\{n \in \mathbb{N} : n \geq 4\} \subseteq X$.

We now give many examples of proofs by induction. For our first example, we establish a formula for the sum of the first n positive natural numbers.

Proposition 2.1.2. *For any $n \in \mathbb{N}^+$, we have*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

i.e.

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

We give two proofs. The first is a clever argument that avoids induction, while the second is a typical application of induction.

Proof 1. We first give a proof without induction. Let $n \in \mathbb{N}^+$ be arbitrary. Let $S = 1 + 2 + \cdots + (n-1) + n$. We also have $S = n + (n-1) + \cdots + 2 + 1$. Adding both of these equalities, we conclude that

$$2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1),$$

and hence

$$2S = n(n+1).$$

Dividing both sides by 2, we conclude that

$$S = \frac{n(n+1)}{2}$$

so $1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$. □

While elegant, the previous argument required the creative insight of rewriting the sum in a different way, and finding the resulting clever pairing. We now give an inductive proof, which replaces the creative leap with some algebraic manipulations.

Proof 2. We give a proof using induction. Since we are proving something about all elements of $\mathbb{N}^+$, we start with a base case of 1.

- *Base Case:* For $n = 1$, the sum on the left-hand side is 1, and the right-hand side is $\frac{1 \cdot 2}{2} = 1$. Thus, that statement is true when $n = 1$.
- *Inductive Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}^+$, i.e. suppose that n is a number for which we know that

  $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

  We then have

  $$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) && \text{(by the inductive hypothesis)} \\ &= \frac{n^2 + n + 2n + 2}{2} \\ &= \frac{n^2 + 3n + 2}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

  Thus, the statement is true for $n + 1$.

By induction, we conclude that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}^+$. □

In the previous proof, we could have written it using the set-theoretic form of induction by letting

$$X = \left\{ n \in \mathbb{N}^+ : \sum_{k=1}^{n} i = \frac{n(n+1)}{2} \right\},$$

and then used the principle of induction to argue that $X = \mathbb{N}^+$. Typically, we will avoid formally writing the set, and working in this way, but it is always possible to translate arguments into the corresponding set-theoretic approach.

Proposition 2.1.3. *For any $n \in \mathbb{N}^+$, we have*

$$\sum_{k=1}^{n}(2k-1) = n^2,$$

i.e.

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2.$$

Proof. We give a proof by induction.

- *Base Case:* Suppose that $n = 1$. We have

  $$\sum_{k=1}^{1}(2k-1) = 2 \cdot 1 - 1 = 1,$$

  so the left hand-side is 1. The right-hand side is $1^2 = 1$. Thus, the statement is true when $n = 1$.

- *Inductive Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}^+$, i.e. suppose that n is a number for which we know that

  $$\sum_{k=1}^{n}(2k-1) = n^2.$$

  Notice that $2(n+1) - 1 = 2n + 2 - 1 = 2n + 1$, hence

  $$\begin{aligned} \sum_{k=1}^{n+1}(2k-1) &= [\sum_{k=1}^{n}(2k-1)] + [2(n+1) - 1] \\ &= [\sum_{k=1}^{n}(2k-1)] + (2n+1) \\ &= n^2 + (2n+1) && \text{(by induction)} \\ &= (n+1)^2. \end{aligned}$$

  Thus, the statement is true for $n + 1$.

By induction, we conclude that

$$\sum_{k=1}^{n}(2k-1) = n^2$$

for all $n \in \mathbb{N}^+$. □

Although induction is a useful tool for proving certain equalities, it can also be used in much more flexible ways. We now give several examples of proving divisibility and inequalities by induction.

Proposition 2.1.4. *For all $n \in \mathbb{N}$, we have $3 \mid (4^n - 1)$.*

Proof. We give a proof by induction.

- *Base Case:* Suppose that $n = 0$. We have $4^0 - 1 = 1 - 1 = 0$, hence $3 \mid (4^0 - 1)$ because $3 \cdot 0 = 0$. Thus, the statement is true when $n = 0$.
- *Inductive Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}^+$, i.e. suppose that n is a number for which we know that $3 \mid (4^n - 1)$. Fix $k \in \mathbb{Z}$ with $3k = 4^n - 1$. Adding 1 to both sides, it follows that $4^n = 3k + 1$, and hence

$$\begin{aligned} 4^{n+1} - 1 &= 4 \cdot 4^n - 1 \\ &= 4 \cdot (3k + 1) - 1 \\ &= 12k + 3 \\ &= 3 \cdot (4k + 1). \end{aligned}$$

  Since $4k + 1 \in \mathbb{Z}$, we conclude that $3 \mid (4^{n+1} - 1)$. Thus, the statement is true for $n + 1$.

By induction, we conclude that $3 \mid (4^n - 1)$ for all $n \in \mathbb{N}$. □

Proposition 2.1.5. *We have $2n + 1 < n^2$ for all $n \in \mathbb{N}$ with $n \geq 3$.*

Proof. We give a proof by induction.

- *Base Case:* Suppose that $n = 3$. We have $2 \cdot 3 + 1 = 7$ and $3^2 = 9$, so $2 \cdot 3 + 1 < 3^2$. Thus, the statement is true when $n = 3$.
- *Inductive Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}$ with $n \geq 3$, i.e. suppose that $n \geq 3$ is a number for which we know that $2n + 1 < n^2$. Since $2n + 1 \geq 2 \cdot 3 + 1 = 7 > 2$, we then have

$$\begin{aligned} 2(n + 1) + 1 &= 2n + 3 \\ &= (2n + 1) + 2 \\ &< n^2 + 2 \\ &< n^2 + 2n + 1 \\ &= (n + 1)^2. \end{aligned}$$

  Thus, the statement is true for $n + 1$.

By induction, we conclude that $2n + 1 < n^2$ for all $n \in \mathbb{N}$ with $n \geq 3$. □

Proposition 2.1.6. *We have $n^2 < 2^n$ for all $n \geq 5$.*

Proof. We give a proof by induction.

- *Base Case:* Suppose that $n = 5$. We have $5^2 = 25$ and $2^5 = 32$, so $5^2 < 2^5$. Thus, the statement is true when $n = 5$.
- *Inductive Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}$ with $n \geq 5$, i.e. suppose that $n \geq 5$ is a number for which we know that $n^2 < 2^n$. Since $n^2 = n \cdot n \geq 3n = 2n + n > 2n + 1$, we have then have

$$\begin{aligned} (n + 1)^2 &= n^2 + 2n + 1 \\ &< n^2 + n^2 \\ &= 2n^2 \\ &< 2 \cdot 2^n \\ &= 2^{n+1}. \end{aligned}$$

Thus, the statement is true for $n+1$.

By induction, we conclude that $n^2 < 2^n$ for all $n \geq 5$. □

Proposition 2.1.7. *For all $x \in \mathbb{R}$ with $x \geq -1$ and all $n \in \mathbb{N}^+$, we hve $(1+x)^n \geq 1+nx$.*

On the face of it, this problem looks a little different because we are also quantifying over infinitely many real numbers x. Since x is coming from $\mathbb{R}$, we can't induct on x. However, we *can* take an arbitrary $x \in \mathbb{R}$ with $x \geq -1$, and then induct on n for this particular x. We now carry out that argument.

Proof. Let $x \in \mathbb{R}$ be arbitrary with $x \geq -1$. For this x, we show that $(1+x)^n \geq 1+nx$ for all $n \in \mathbb{N}^+$ by induction.

- *Base Case:* Suppose that $n = 1$. We then have that $(1+x)^1 = 1+x = 1+1x$, so certainly $(1+x)^1 \geq 1+1x$.
- *Inductive Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}^+$, i.e. suppose that n is a number for which we know that $(1+x)^n \geq 1+nx$. Since $x \geq -1$, we have $1+x \geq 0$, so we can multiply both sides of this inequality by $(1+x)$ to conclude that

$$(1+x)^n \cdot (1+x) \geq (1+nx) \cdot (1+x).$$

We then have

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n \cdot (1+x) \\
&\geq (1+nx) \cdot (1+x) && \text{(from above)} \\
&= 1+nx+x+nx^2 \\
&= 1+(n+1)x+nx^2 \\
&\geq 1+(n+1)x. && \text{(since } nx^2 \geq 0\text{)}
\end{aligned}$$

Hence, we have shown that $(1+x)^{n+1} \geq 1+(n+1)x$, i.e. that the statement is true for $n+1$.

By induction, we conclude that $(1+x)^n \geq 1+nx$ for all $n \in \mathbb{N}^+$. Since $x \in \mathbb{R}$ with $x \geq -1$ was arbitrary, the result follows. □

Proposition 2.1.8. *For all $n \in \mathbb{N}^+$, we have*

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

Proof. We prove the statement by induction.

- *Base Case:* Suppose that $n = 1$. In this case, we have

$$\sum_{k=1}^{1} \frac{1}{k^2} = \frac{1}{1^2} = 1$$

and

$$2 - \frac{1}{1} = 2 - 1 = 1$$

hence

$$\sum_{k=1}^{1} \frac{1}{k^2} \leq 2 - \frac{1}{1}.$$

- *Inductive Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}^+$, i.e. suppose that n is a number for which we know that

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

We then have

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k^2} &= \left(\sum_{k=1}^{n} \frac{1}{k^2} \right) + \frac{1}{(n+1)^2} \\
&\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\
&= 2 - \left(\frac{1}{n} - \frac{1}{(n+1)^2} \right) \\
&= 2 - \frac{(n+1)^2 - n}{n(n+1)^2} \\
&= 2 - \frac{n^2 + n + 1}{n(n+1)^2} \\
&\leq 2 - \frac{n^2 + n}{n(n+1)^2} \\
&= 2 - \frac{n(n+1)}{n(n+1)^2} \\
&= 2 - \frac{1}{n+1}.
\end{aligned}$$

Thus, the statement is true for $n + 1$.

By induction, we conclude we conclude that

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

for all $n \in \mathbb{N}^+$. □

2.2 Strong Induction and Well-Ordering

Remember our original model for induction:

- Prove that the statement is true for 0.
- Prove that if the statement is true for 0, then the statement is true for 1.
- Prove that if the statement is true for 1, then the statement is true for 2.
- Prove that if the statement is true for 2, then the statement is true for 3.
- Prove that if the statement is true for 3, then the statement is true for 4.
- ....

In the previous section, we argued why this model was sound and gave many examples. However, upon closer inspection, it appears that we can assume more. In the second line, when proving that the statement is true for 1 we are allowed to assume that the statement is true for 0. Now in the third line, when proving that the statement is true for 2, we only assume that it is true for 1. If we are knocking down the natural numbers in order, then we've already proved that it's true for 0, so why can't we assume that as well? The answer is that we can indeed assume it! In general, when working to prove that the statement is true for a natural number n, we can assume that the statement is true for all smaller natural numbers. In other words, we do the following:

- Prove that the statement is true for 0.
- Prove that if the statement is true for 0, then the statement is true for 1.
- Prove that if the statement is true for 0 and 1, then the statement is true for 2.
- Prove that if the statement is true for 0, 1, and 2, then the statement is true for 3.
- Prove that if the statement is true for 0, 1, 2, and 3, then the statement is true for 4.
- ....

Suppose that we are successful in doing this. From the first line, we then know that the statement is true for 0. Since we now know that it's true for 0, we can use the second line to conclude that the statement is true for 1. Since we now know that it's true for both 0 and 1, we can use the second line to conclude that the statement is true for 2. And so on. In the end, we are able to conclude that the statement is true for all natural numbers.

As usual, we can't hope to prove each of these infinitely many things one at a time. In an ideal world, the arguments from the second line onward all look exactly the same with the exception of replacing the number involved. Thus, the idea is to prove the following.

- Prove that the statement is true for 0.
- Prove that if the statement is true for each of $0, 1, 2, \ldots, n$, then the statement is true for $n + 1$.

Alternatively, we can state this as follows:

- Prove that the statement is true for 0.
- Prove that if the statement is true for each of $0, 1, 2, \ldots, n - 1$, then the statement is true for n (for $n \geq 1$).

An argument using these method is called a proof by *strong induction*. As we will see in the examples below, sometimes we need to modify this simple structure to include several base cases in order to get the argument going. Rather than going through a theoretical discussion of how and why one would do this, it's easier to illustrate the technique by example.

We start with an example where we verify a simple closed formed formula for a recursively defined sequence. Since the sequence uses the past two values to define the current value, regular induction does not give enough power to complete the proof.

Proposition 2.2.1. *Define a sequence a_n recursively by letting $a_0 = 0$, $a_1 = 1$, and*

$$a_n = 3a_{n-1} - 2a_{n-2}$$

for $n \geq 2$. Show that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Proof. We prove that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$ by strong induction.

- *Base Case:* We handle two bases where $n = 0$ and $n = 1$ because our inductive step will use the result for two steps back. When $n = 0$, we have $a_0 = 0$ and $2^0 - 1 = 1 - 1 = 0$, so $a_0 = 2^0 - 1$. When $n = 1$, we have $a_1 = 1$ and $2^1 - 1 = 2 - 1 = 1$, so $a_1 = 2^1 - 1$.

- *Inductive Step:* Let $n \geq 2$ and assume that the statement is true for $0, 1, 2, \ldots, n-1$, i.e. assume that $a_m = 2^m - 1$ for all $m \in \{0, 1, 2, \ldots, n-1\}$. We prove that the statement is true for n. Notice that since $n \geq 2$, we have $0 \leq n - 1 < n$ and $0 \leq n - 2 < n$, so we know that $a_{n-1} = 2^{n-1} - 1$ and $a_{n-2} = 2^{n-2} - 1$. Now

$$\begin{aligned} a_n &= 3a_{n-1} - 2a_{n-1} && \text{(by definition since } n \geq 2) \\ &= 3 \cdot (2^{n-1} - 1) - 2 \cdot (2^{n-2} - 1) && \text{(by the inductive hypothesis)} \\ &= 3 \cdot 2^{n-1} - 3 - 2 \cdot 2^{n-2} + 2 \\ &= 3 \cdot 2^{n-1} - 2^{n-1} - 1 \\ &= (3 - 1) \cdot 2^{n-1} - 1 \\ &= 2 \cdot 2^{n-1} - 1 \\ &= 2^n - 1. \end{aligned}$$

  Thus, $a_n = 2^n - 1$ and so the statement is true for n.

Using strong induction, we conclude that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$. □

We now turn to an interesting example of using strong induction to establish when we can solve an equation in the natural numbers.

Proposition 2.2.2. *If $n \in \mathbb{N}$ and $n \geq 12$, then there exist $k, \ell \in \mathbb{N}$ with $n = 3k + 7\ell$.*

Proof. We give a proof by strong induction.

- *Base Case:* We first prove that the statement is true for all $n \in \{12, 13, 14\}$ (we will see why we need so many base cases in the inductive step below). We have the following cases:
  - $12 = 3 \cdot 4 + 7 \cdot 0$.
  - $13 = 3 \cdot 2 + 7 \cdot 1$.
  - $14 = 3 \cdot 0 + 7 \cdot 2$.

  Thus, the statement is true for all $n \in \{12, 13, 14\}$.

- *Inductive Step:* Let $n \geq 15$ and assume that the statement is true for all $k \in \mathbb{N}$ with $12 \leq k < n$, i.e. assume that the statement is true for $12, 13, 14, \ldots, n-1$. We prove that the statement is true for n. Since $n \geq 15$, we have $12 \leq n - 3 < n$, so we can use the inductive hypothesis to fix $k, \ell \in \mathbb{N}$ with

$$n - 3 = 3k + 7\ell.$$

  Adding 3 to both sides, we see that

$$\begin{aligned} n &= 3k + 7\ell + 3 \\ &= 3(k + 1) + 7\ell. \end{aligned}$$

  Since $k + 1, \ell \in \mathbb{N}$, we conclude that the statement is true for n.

By strong induction, we conclude that for all $n \in \mathbb{N}$ with $n \geq 12$, there exist $k, \ell \in \mathbb{N}$ with $n = 3k + 7\ell$. □

We can also use strong induction to establish bounds for recursively defined sequences.

Proposition 2.2.3. *Define a sequence recursively by letting $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. We have*

$$f_n \leq 2^n$$

for all $n \in \mathbb{N}$.

Proof. We prove the result by strong induction.

- *Base Case:* We first handle the cases when $n = 0$ and $n = 1$.
  - Notice that $2^0 = 1 > 0$, so $f_0 \leq 2^0$.
  - Notice that $2^1 = 2 > 1$, so $f_1 \leq 2^1$.

  Thus, the statement is true for $n = 0$ and $n = 1$.
- *Inductive Step:* Suppose that $n \geq 2$ and the statement is true for all $k \in \mathbb{N}$ with $k < n$. In particular, we have $0 \leq n - 2 < n$ and $0 \leq n - 1 < n$, so the statement is true for both $n - 2$ and $n - 1$, and hence $f_{n-2} \leq 2^{n-2}$ and $f_{n-1} \leq 2^{n-1}$. We then have

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} && \text{(since } n \geq 2\text{)} \\ &\leq 2^{n-1} + 2^{n-2} && \text{(from above)} \\ &\leq 2^{n-1} + 2^{n-1} \\ &= 2 \cdot 2^{n-1} \\ &= 2^n. \end{aligned}$$

  Therefore, $f_n \leq 2^n$, i.e. the statement is true for n.

By strong induction, we conclude that $f_n \leq 2^n$ for all $n \in \mathbb{N}$. □

Once we have such a proof, it is natural to ask how it could be improved. A nearly identical argument shows that $f_n \leq 2^{n-1}$ for all $n \in \mathbb{N}$. However, if we try to show that $f_n \leq 2^{n-2}$ for all $n \in \mathbb{N}$, then the inductive step goes through without a problem, but the base case of $n = 1$ does not work. As a result, the argument fails.

Can we obtain a significantly better upper bound for f_n than 2^{n-1}? In particular, can we use an exponential whose base is less than 2? If we replace 2 with a number $\alpha > 1$, i.e. try to prove that $f_n \leq \alpha^n$ (or $f_n \leq \alpha^{n-1}$), then the base case goes through without a problem. In the inductive step, the key fact that we used was that $2^{n-1} + 2^{n-2} \leq 2^n$ for all $n \in \mathbb{N}$. If we replace 2 by an $\alpha > 1$ with the property that $\alpha^{n-1} + \alpha^{n-2} \leq \alpha^n$ for all $n \in \mathbb{N}$, then we can carry out the argument. Dividing through by α^{n-2}, we want to find the smallest possible $\alpha > 1$ such that $\alpha + 1 \leq \alpha^2$, which is equivalent to $\alpha^2 - \alpha - 1 \geq 0$. Using the quadratic formula, the solutions to $x^2 - x - 1 = 0$ are

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

Now $\frac{1+\sqrt{5}}{2} > 1$, so we now go back and check that we can use it in an inductive argument. In fact, we can use it as a lower bound too (due to the fact that we get *equality* at the necessary step), so long as we change the exponent slightly and start with f_1.

Proposition 2.2.4. *Define a sequence recursively by letting $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Let $\phi = \frac{1+\sqrt{5}}{2}$ and notice that $\phi^2 = \phi + 1$ (either from above, or by direct calculation). We have*

$$\phi^{n-2} \leq f_n \leq \phi^{n-1}$$

for all $n \in \mathbb{N}^+$.

Proof. We prove the result by strong induction.

- *Base Case:* We first handle the cases when $n = 1$ and $n = 2$. Notice that

$$\phi = \frac{1+\sqrt{5}}{2} > \frac{1+2}{2} = \frac{3}{2},$$

hence

$$\phi^{-1} < \frac{2}{3}.$$

We also have

$$\phi = \frac{1+\sqrt{5}}{2} < \frac{1+3}{2} = 2.$$

Since $f_1 = 1 = f_2$, we have

$$\phi^{-1} < f_1 = \phi^0$$

and

$$\phi^0 = f_2 < \phi^1.$$

Therefore, the statement is true for $n = 1$ and $n = 2$.

- *Inductive Step:* Suppose that $n \geq 3$ and the statement is true for all $k \in \mathbb{N}^+$ with $k < n$. In particular, we have $1 \leq n-2 < n$ and $1 \leq n-1 < n$, so the statement is true for both $n-2$ and $n-1$, and hence

$$\phi^{n-4} \leq f_{n-2} \leq \phi^{n-3} \qquad \text{and} \qquad \phi^{n-3} \leq f_{n-1} \leq \phi^{n-2}$$

We have

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} && \text{(since } n \geq 3\text{)} \\ &\geq \phi^{n-3} + \phi^{n-4} && \text{(from above)} \\ &= \phi^{n-4}(\phi + 1) \\ &= \phi^{n-4} \cdot \phi^2 \\ &= \phi^{n-2}, \end{aligned}$$

and also

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} && \text{(since } n \geq 3\text{)} \\ &\leq \phi^{n-2} + \phi^{n-3} && \text{(from above)} \\ &= \phi^{n-3}(\phi + 1) \\ &= \phi^{n-3} \cdot \phi^2 \\ &= \phi^{n-1}. \end{aligned}$$

Therefore, $\phi^{n-2} \leq f_n \leq \phi^{n-1}$, i.e. the statement is true for n.

By strong induction, we conclude that $\phi^{n-2} \leq f_n \leq \phi^{n-1}$ for all $n \in \mathbb{N}^+$. □

Closely related to strong induction, the following is a core fact about the ordering of the natural numbers:

Fact 2.2.5 (Well-Ordering of $\mathbb{N}$)**.** *Every nonempty set $X \subseteq \mathbb{N}$ has a smallest element. That is, for all nonempty $X \subseteq \mathbb{N}$, there exists $m \in X$ such that $m \leq n$ for all $n \in X$.*

Why is this statement true? Suppose that $X \subseteq \mathbb{N}$ is nonempty. If $0 \in X$, then 0 is clearly the smallest element of X, and we are done. Suppose then that $0 \notin X$. If $1 \in X$, then 1 is the smallest element of X, and we are done. Suppose then that $1 \notin X$. If $2 \in X$, then 2 is the smallest element of X, and we are done. Continuing this process, we must eventually reach a point where we encounter an element X, because otherwise we would eventually argue that each fixed $n \in \mathbb{N}$ is not an element of X, which would then imply that $X = \emptyset$.

This argument, like the arguments for induction and strong induction, is intuitively reasonable and convincing. However, it is not particularly formal. It is possible to formally prove each of induction, strong induction, and well-ordering from any of the others, so in a certain precise sense the three statements are equivalent. If you're interested, think about how to prove well-ordering using induction (along with some of the other implications). However, since all three are intuitively very reasonable, and it's beyond the scope of the course to construct the natural numbers and articulate exactly what we are allowed to use in the proofs of these equivalences, we will omit the careful arguments.

Notice that the given statement is false if we consider subsets of $\mathbb{Z}$ or $\mathbb{R}$ (rather than subsets of $\mathbb{N}$). For example, $\mathbb{Z}$ is trivially a nonempty subset of $\mathbb{Z}$, but it does not have a smallest element. Even if we consider only subsets of the nonnegative reals $\{x \in \mathbb{R} : x \geq 0\}$, we can find nonempty subsets with no smallest element (for example, the open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ does not have a smallest element).

We can often write an inductive proof as a proof using well-ordering, by considering a smallest potential counterexample. For example, here is a proof of Proposition 2.2.2 (if $n \in \mathbb{N}$ and $n \geq 12$, then there exist $k, \ell \in \mathbb{N}$ with $n = 3k + 7\ell$) using a well-ordering argument.

Proof of Proposition 2.2.2. Consider the set

$$X = \{n \in \mathbb{N} : n \geq 12 \text{ and there does not exist } k, \ell \in \mathbb{N} \text{ with } n = 3k + 7\ell\}$$

of counterexamples to the given statement. It suffices to show that $X = \emptyset$. Suppose instead that $X \neq \emptyset$. By well-ordering, we can let m be the smallest element of X. Notice that $m \notin \{12, 13, 14\}$ because we have the following:

- $12 = 3 \cdot 4 + 7 \cdot 0$.
- $13 = 3 \cdot 2 + 7 \cdot 1$.
- $14 = 3 \cdot 0 + 7 \cdot 2$.

Therefore, we must have $m \geq 15$, and hence $12 \leq m - 3 < 15$. Now m is the smallest element of X, so we must have $m - 3 \notin X$, and hence we can fix $k, \ell \in \mathbb{N}$ with $m - 3 = 3k + 7\ell$. Adding 3 to both sides, we see that

$$\begin{aligned} m &= 3k + 7\ell + 3 \\ &= 3(k + 1) + 5\ell. \end{aligned}$$

Since $k + 1, \ell \in \mathbb{N}$, we conclude that $m \notin X$, which is a contradiction. Therefore, it must be the case that $X = \emptyset$, giving the result. □

2.3 Division with Remainder

The primary goal of this section is to prove the following deeply fundamental result.

Theorem 2.3.1. *Let $a, b \in \mathbb{Z}$ with $b \neq 0$. There exist unique $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < |b|$. Uniqueness here means that if $a = q_1 b + r_1$ with $0 \leq r_1 < |b|$ and $a = q_2 b + r_2$ with $0 \leq r_2 < |b|$, then $q_1 = q_2$ and $r_1 = r_2$.*

Here are a bunch of examples illustrating existence:

- If $a = 5$ and $b = 2$, then we have $5 = 2 \cdot 2 + 1$.
- If $a = 135$ and $b = 45$, then we have $135 = 3 \cdot 45 + 0$.
- If $a = 60$ and $b = 9$, then we have $60 = 6 \cdot 9 + 6$.
- If $a = 29$ and $b = -11$, then we have $29 = (-2)(-11) + 7$.
- If $a = -45$ and $b = 7$, then we have $-45 = (-7) \cdot 7 + 4$.
- If $a = -21$ and $b = -4$, then we have $-21 = 6 \cdot (-4) + 3$.

We begin by proving existence via a sequence of lemmas, starting in the special case where a and b are both natural numbers.

Lemma 2.3.2. *Let $a, b \in \mathbb{N}$ with $b > 0$. There exist $q, r \in \mathbb{N}$ such that $a = qb + r$ and $0 \leq r < b$.*

We give three separate proofs (induction, strong induction, and well-ordering) to illustrate the different perspectives.

Proof 1 of Lemma 2.3.2 - By Induction. Let $b \in \mathbb{N}$ with $b > 0$ be arbitrary. For this fixed b, we prove the existence of both q and r for all $a \in \mathbb{N}$ by induction. That is, for this fixed b, we define

$$X = \{a \in \mathbb{N} : \text{ There exist } q, r \in \mathbb{N} \text{ with } a = qb + r\},$$

and show that $X = \mathbb{N}$ by induction.

- *Base Case:* Suppose that $a = 0$. We then have $a = 0 \cdot b + 0$ and clearly $0 < b$, so we may take $q = 0$ and $r = 0$.
- *Inductive Step:* Let $a \in \mathbb{N}$ be arbitrary such that $a \in X$. We show that $a + 1 \in X$. Since $a \in X$, we can fix $q, r \in \mathbb{N}$ with $0 \leq r < b$ such that $a = qb + r$. We then have $a + 1 = qb + (r + 1)$. Since $b, r \in \mathbb{N}$ and $r < b$, we know that $r + 1 \leq b$. If $r + 1 < b$, then we are done. Otherwise, we have $r + 1 = b$, hence

$$\begin{aligned} a + 1 &= qb + (r + 1) \\ &= qb + b \\ &= (q + 1)b \\ &= (q + 1)b + 0, \end{aligned}$$

  so we may take $q + 1$ and 0.

By induction, we conclude that $X = \mathbb{N}$. Since b was arbitrary, the result follows. □

Proof 2 of Lemma 2.3.2 - By Strong Induction. Let $b \in \mathbb{N}$ with $b > 0$ be arbitrary. For this fixed b, we prove the existence of both q and r for all $a \in \mathbb{N}$ by strong induction. That is, for this fixed b, we define

$$X = \{a \in \mathbb{N} : \text{ There exist } q, r \in \mathbb{N} \text{ with } a = qb + r\},$$

and show that $X = \mathbb{N}$ by strong induction.

- *Base Case:* Let $a \in \mathbb{N}$ with $a < b$ be arbitrary. We then have $a = 0 \cdot b + a$ and clearly $a < b$, so we may take $q = 0$ and $r = a$.

- *Inductive Step:* Let $a \in \mathbb{N}$ with $a \geq b$ be arbitrary, and assume that $c \in X$ for all $c \in \mathbb{N}$ with $c < a$. We show that $a \in X$. Since $a \geq b$, we can subtract b from both sides to conclude that $a - b \geq 0$, and hence $a - b \in \mathbb{N}$. Also, since $b > 0$, we know that $a - b < a$. Since we have $0 \leq a - b < a$, we know that $a - b \in X$, so we can fix $q, r \in \mathbb{N}$ with $0 \leq r < b$ such that $a - b = qb + r$. Adding b to both sides, it follows that

$$\begin{aligned} a &= qb + r + b \\ &= qb + b + r \\ &= (q+1)b + r, \end{aligned}$$

  so we may take $q + 1$ and r.

By strong induction, we conclude that $X = \mathbb{N}$. Since b was arbitrary, the result follows. □

Proof 3 of Lemma 2.3.2 - By Well-Ordering. Let $a, b \in \mathbb{N}$ with $b > 0$ be arbitrary. Consider the set

$$S = \{a - kb : k \in \mathbb{N}\} \cap \mathbb{N}.$$

Notice that $a \in S$ (by taking $k = 0$ and recalling that $a \in \mathbb{N}$), so $S \neq \emptyset$. By well-ordering, S has a smallest element $r \in \mathbb{N}$. Since $r \in S$, we can fix $q \in \mathbb{N}$ with $r = a - qb$. We then have that $a = qb + r$, so we need only show that $r < b$. Notice that

$$\begin{aligned} r - b &= a - qb - b \\ &= a - (q+1)b, \end{aligned}$$

so as $q + 1 \in \mathbb{N}$, it follows that $r - b \in \{a - kb : k \in \mathbb{N}\}$. Now $r - b < r$ because $b > 0$, so as r is the smallest element of S, it must be the case that $r - b \notin S$. As a result, we conclude that $r - b \notin \mathbb{N}$, so must have $r - b < 0$ (because clearly $r - b \in \mathbb{Z}$). Adding b to both sides, it follows that $r < b$. □

With this in hand, we now extend to the case where $a \in \mathbb{Z}$.

Lemma 2.3.3. *Let $a, b \in \mathbb{Z}$ with $b > 0$. There exist $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < b$.*

Proof. If $a \geq 0$, we are done by the previous lemma. Suppose that $a < 0$. We then have $-a > 0$, so by the previous lemma we may fix $q, r \in \mathbb{N}$ with $0 \leq r < b$ such that $-a = qb + r$. We then have $a = -(qb + r) = (-q)b + (-r)$. If $r = 0$, then $-r = 0$ and we are done. Otherwise we $0 < r < b$ and

$$\begin{aligned} a &= (-q)b + (-r) \\ &= (-q)b - b + b + (-r) \\ &= (-q-1)b + (b - r). \end{aligned}$$

Now since $0 < r < b$, we have $0 < b - r < b$, so this gives existence. □

And now we can extend to the case where $b < 0$.

Lemma 2.3.4. *Let $a, b \in \mathbb{Z}$ with $b \neq 0$. There exist $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < |b|$.*

Proof. If $b > 0$, we are done by the previous lemma. Suppose that $b < 0$. We then have $-b > 0$, so by the previous lemma we can fix $q, r \in \mathbb{N}$ with $0 \leq r < -b$ and $a = q(-b) + r$. We then have $a = (-q)b + r$ and we are done because $|b| = -b$. □

With that sequence of lemmas building to existence now in hand, we finish off the proof of the theorem.

Proof of Theorem 2.3.1. The final lemma above gives us existence, so we need only prove uniqueness. Let $q_1, r_1, q_2, r_2 \in \mathbb{Z}$ be arbitrary with

$$q_1 b + r_1 = a = q_2 b + r_2,$$

and where $0 \leq r_1 < |b|$ and $0 \leq r_2 < |b|$. We need to show that $q_1 = q_2$ and $r_1 = r_2$. Manipulating the above equality, we have

$$b(q_2 - q_1) = r_1 - r_2,$$

hence $b \mid (r_2 - r_1)$. Now $-|b| < -r_1 \leq 0$, so adding this to $0 \leq r_2 < |b|$, we conclude that

$$-|b| < r_2 - r_1 < |b|,$$

and therefore

$$|r_2 - r_1| < |b|.$$

Now if $r_2 - r_1 \neq 0$, then since $b \mid (r_2 - r_1)$, we can use Proposition 1.5.4 to conclude that $|b| \leq |r_2 - r_1|$, a contradiction. It follows that $r_2 - r_1 = 0$, and hence $r_1 = r_2$. Since

$$q_1 b + r_1 = q_2 b + r_2$$

and $r_1 = r_2$, we conclude that $q_1 b = q_2 b$. Now $b \neq 0$, so we can divide both sides by b to conclude that $q_1 = q_2$. □

Now that we have established the core facts about division with remainder, we can use them to give a simple check for divisibility.

Proposition 2.3.5. *Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Write $a = qb + r$ for the unique choice of $q, r \in \mathbb{Z}$ with $0 \leq r < |b|$. We then have that $b \mid a$ if and only if $r = 0$.*

Proof. If $r = 0$, then $a = qb + r = bq$, so $b \mid a$. Suppose conversely that $b \mid a$ and fix $m \in \mathbb{Z}$ with $a = bm$. We then have $a = mb + 0$ and $a = qb + r$, so by the uniqueness part of the above theorem, we must have $r = 0$. □

For example, we can now easily verify that $2 \nmid 5$ without any work as follows. Simply notice that $5 = 2 \cdot 2 + 1$ and $0 \leq 1 < 2$, so since the unique remainder is $1 \neq 0$, it follows that $2 \nmid 5$.

Chapter 3

GCDs, Primes, and the Fundamental Theorem of Arithmetic

3.1 The Euclidean Algorithm

Definition 3.1.1. *Suppose that $a, b \in \mathbb{Z}$. We say that $d \in \mathbb{Z}$ is a* common divisor *of a and b if both $d \mid a$ and $d \mid b$.*

We can write the set of common divisors of a and b as an intersection, i.e. given $a, b \in \mathbb{Z}$, the set of common divisors of a and b is the set $Div(a) \cap Div(b)$. For example, the set of common divisors of 120 and 84 is the set $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$. One way to determine the values in this set is to exhaustively determine each of the sets $Div(120)$ and $Div(84)$, and then comb through them both to find the common elements. However, we will work out a much more efficient way to solve such problems in this section.

The set of common divisors of 10 and 0 is $\{\pm 1, \pm 2, \pm 5, \pm 10\}$ because $Div(0) = \mathbb{Z}$, and hence the set of common divisors of 10 and 0 is just $Div(10) \cap Div(0) = Div(10) \cap \mathbb{Z} = Div(10)$. In contrast, every element of $\mathbb{Z}$ is a common divisor of 0 and 0, because $Div(0) \cap Div(0) = \mathbb{Z} \cap \mathbb{Z} = \mathbb{Z}$. The following little proposition is fundamental to this entire section.

Proposition 3.1.2. *Suppose that $a, b, q, r \in \mathbb{Z}$ and $a = qb + r$ (we do* not *assume that $0 \leq r < |b|$). We then have $Div(a) \cap Div(b) = Div(b) \cap Div(r)$, i.e.*

$$\{d \in \mathbb{Z} : d \text{ is a common divisor of } a \text{ and } b\} = \{d \in \mathbb{Z} : d \text{ is a common divisor of } b \text{ and } r\}.$$

Proof. Let $d \in Div(b) \cap Div(r)$ be arbitrary. Since $d \mid b$, $d \mid r$, and $a = qb + r = q \cdot b + 1 \cdot r$, we may use Proposition 1.5.3 to conclude that $d \mid a$. Therefore, $d \in Div(a) \cap Div(b)$.

Conversely, let $d \in Div(a) \cap Div(b)$ be arbitrary. Since $d \mid a$, $d \mid b$, and $r = a - qb = 1 \cdot a + (-q) \cdot b$, we may use Proposition 1.5.3 to conclude that $d \mid r$. Therefore, $d \in Div(b) \cap Div(r)$. □

For example, suppose that we are trying to find the set of common divisors of 120 and 84, i.e. we want to understand the elements of the set $Div(120) \cap Div(84)$ (we wrote them above, but now want to justify it). We repeatedly perform division with remainder to reduce the problem as follows:

$$\begin{aligned} 120 &= 1 \cdot 84 + 36 \\ 84 &= 2 \cdot 36 + 12 \\ 36 &= 3 \cdot 12 + 0. \end{aligned}$$

The first line tells us that

$$Div(120) \cap Div(84) = Div(84) \cap Div(36).$$

The next line tells us that

$$Div(84) \cap Div(36) = Div(36) \cap Div(12).$$

The last line tells us that

$$Div(36) \cap Div(12) = Div(12) \cap Div(0).$$

Now $Div(0) = \mathbb{Z}$, so

$$Div(12) \cap Div(0) = Div(12).$$

Putting it all together, we conclude that

$$Div(120) \cap Div(84) = Div(12),$$

which is a more elegant way to determine the set of common divisors of 120 and 84 than the exhaustive process we alluded to above.

The above arguments illustrates the idea behind the following very general and important fact:

Theorem 3.1.3. *For all $a, b \in \mathbb{Z}$, there exists a unique $m \in \mathbb{N}$ such that $Div(a) \cap Div(b) = Div(m)$. In other words, for any $a, b \in \mathbb{Z}$, we can always find a (unique) natural number m such that the set of common divisors of a and b equals the set of divisors of m.*

We first sketch the idea of the proof of existence in the case where $a, b \in \mathbb{N}$. If $b = 0$, then since $Div(0) = \mathbb{Z}$, we can simply take $m = a$. Suppose then that $b \neq 0$. Fix $q, r \in \mathbb{N}$ with $a = qb + r$ and $0 \leq r < b$. Now the idea is to inductively assert the existence of an m that works for the pair of numbers (b, r) because this pair is "smaller" than the pair (a, b). The only issue is how to make this intuitive idea of "smaller" precise. There are several ways to do this, but perhaps the most straightforward is to only induct on b. Thus, our base case handles all pairs of form $(a, 0)$. Next, we handle all pairs of the form $(a, 1)$ and in doing this we can use the fact the we know the result for all pairs of the form $(a', 0)$. Notice that we can we even change the value of the first coordinate here, which is why we used the notation a'. Then, we handle all pairs of the form $(a, 2)$ and in doing this we can use the fact that we know the result for all pairs of the form $(a', 0)$ and $(a', 1)$. We now carry out the formal argument.

Proof. We begin by proving existence only in the special case where $a, b \in \mathbb{N}$. We use (strong) induction on b to prove the result. That is, we let

$$X = \{b \in \mathbb{N} : \text{For all } a \in \mathbb{N}, \text{ there exists } m \in \mathbb{N} \text{ with } Div(a) \cap Div(b) = Div(m)\}$$

and prove that $X = \mathbb{N}$ by strong induction.

- *Base Case:* Suppose that $b = 0$. Let $a \in \mathbb{N}$ be arbitrary. We then have that $Div(b) = \mathbb{Z}$, so

  $$Div(a) \cap Div(b) = Div(a) \cap \mathbb{Z} = Div(a),$$

  and hence we may take $m = a$. Since $a \in \mathbb{N}$ was arbitrary, we showed that $0 \in X$.

- *Inductive Step:* Let $b \in \mathbb{N}^+$ be arbitrary, and suppose that we know that the statement is true for all smaller natural numbers. In other words, we are assuming that $c \in X$ whenever $0 \leq c < b$. We prove that $b \in X$. Let $a \in \mathbb{N}$ be arbitrary. From above, we may fix $q, r \in \mathbb{Z}$ with $a = qb + r$ and $0 \leq r < b$. Since $0 \leq r < b$, we know by strong induction that $r \in X$, so we can fix $m \in \mathbb{N}$ with

  $$Div(b) \cap Div(r) = Div(m).$$

  By Proposition 3.1.2, we have that $Div(a) \cap Div(b) = Div(b) \cap Div(r)$. Therefore, $Div(a) \cap Div(b) = Div(m)$. Since $a \in \mathbb{N}$ was arbitrary, we showed that $b \in X$.

Therefore, we have shown that $X = \mathbb{N}$, which implies that whenever $a, b \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $Div(a) \cap Div(b) = Div(m)$.

To prove the existence statement more generally when $a, b \in \mathbb{Z}$, we use Proposition 1.5.7. So, for example, if $a < 0$ but $b \geq 0$, we can fix $m \in \mathbb{N}$ with $Div(-a) \cap Div(b) = Div(m)$, and then use the fact that $Div(a) = Div(-a)$ to conclude that $Div(a) \cap Div(b) = Div(m)$. A similar argument works if $a \geq 0$ and $b < 0$, or if both $a < 0$ and $b < 0$.

For uniqueness, suppose that $m, n \in \mathbb{N}$ are such that both $Div(a) \cap Div(b) = Div(m)$ and also $Div(a) \cap Div(b) = Div(n)$. We then have that $Div(m) = Div(n)$. Since $m \in Div(m)$ trivially, we have that $m \in Div(n)$, so $m \mid n$. Similarly, we have $n \mid m$. Therefore, by Corollary 1.5.5, either $m = n$ or $m = -n$. Since $m, n \in \mathbb{N}$, we have $m \geq 0$ and $n \geq 0$, so it must be the case that $m = n$. □

With this great result in hand, we now turn our attention to a fundamental concept: greatest common divisors. Given $a, b \in \mathbb{Z}$, one might be tempted to define the greatest common divisor of a and b to be the *largest* natural number that divides both a and b (after all, the name *greatest* surely suggests this!). However, it turns out that this is a poor definition for several reasons:

1. Consider the case $a = 120$ and $b = 84$ from above. We saw that the set of common divisors of 120 and 84 is $Div(120) \cap Div(84) = Div(12)$. Thus, the *largest* natural number that divides both 120 and 84 is 12, but in fact 12 has a much stronger property: *every* common divisor of 120 and 84 is also a divisor of 12. This stronger property is surprising and much more fundamental that simply being the largest common divisor.

2. There is one pair of integers where no largest common divisors exists! In the trivial case where $a = 0$ and $b = 0$, *every* integer is a common divisor of a and b. Although this is a somewhat silly edge case, we would ideally like a definition that handles all cases elegantly.

3. The integers have a natural ordering associated with them, but we will eventually want to generalize the idea of a greatest common divisor to settings where there is no analogue of $\leq$ (see Abstract Algebra).

With all of this background in mind, we now give our formal definition.

Definition 3.1.4. *Let $a, b \in \mathbb{Z}$. We say that an element $m \in \mathbb{Z}$ is a* greatest common divisor *of a and b if the following are all true:*

- $m \geq 0$.
- *m is a common divisor of a and b.*
- *Whenever $d \in \mathbb{Z}$ is a common divisor of a and b, we have $d \mid m$.*

In other words, a greatest common divisor is a natural number, is a common divisor, and has the property that every common divisors happens to divide it. In terms of point 3 above, it is a straightforward matter to check that 0 is in fact a greatest common divisor of 0 and 0, because every element of $Div(0) \cap Div(0) = \mathbb{Z}$ is a divisor of 0.

Since we require more of a greatest common divisor than just picking the largest, we first need to check that they do indeed exist. However, the next proposition reduces this task to our previous work.

Proposition 3.1.5. *Let $a, b \in \mathbb{Z}$ and let $m \in \mathbb{N}$. The following are equivalent:*

1. *$Div(a) \cap Div(b) = Div(m)$.*
2. *m is a greatest common divisor of a and b.*

Proof. We first prove (1) $\Rightarrow$ (2): Suppose then that $Div(a) \cap (b) = Div(m)$. Since we are assuming that $m \in \mathbb{N}$, we have that $m \geq 0$. Since $m \mid m$, we have $m \in Div(m)$, so $m \in Div(a) \cap Div(b)$, and hence both $m \mid a$ and $m \mid b$. Now let $d \in \mathbb{Z}$ be an arbitrary common divisor of a and b. We then have that both $d \mid a$ and $d \mid b$, so $d \in Div(a) \cap Div(b)$, hence $d \in Div(m)$, and therefore $d \mid m$. Putting it all together, we conclude that m is a greatest common divisor of a and b.

We now prove (2) $\Rightarrow$ (1): Suppose that m is a greatest common divisor of a and b. We need to prove that $Div(a) \cap Div(b) = Div(m)$.

- We first show that $Div(a) \cap Div(b) \subseteq Div(m)$. Let $d \in Div(a) \cap Div(b)$ be arbitrary. We then have both $d \mid a$ and $d \mid b$, so since m is a greatest common divisor of a and b, we conclude that $d \mid m$. Therefore, $d \in Div(m)$.
- We now show that $Div(m) \subseteq Div(a) \cap Div(b)$. Let $d \in Div(m)$ be arbitrary, so $d \mid m$. Now we know that m is a common divisor of a and b, so both $m \mid a$ and $m \mid b$. Using Proposition 1.5.2, we conclude that both $d \mid a$ and $d \mid b$, so $d \in Div(a) \cap Div(b)$.

Since we have shown both $Div(a) \cap Div(b) \subseteq Div(m)$ and $Div(m) \subseteq Div(a) \cap Div(b)$, we conclude that $Div(a) \cap Div(b) \subseteq Div(m)$. □

Corollary 3.1.6. *Every pair of integers $a, b \in \mathbb{Z}$ has a unique greatest common divisor.*

Proof. Immediate from Theorem 3.1.3 and Proposition 3.1.5. □

Definition 3.1.7. *Let $a, b \in \mathbb{Z}$. We let $\gcd(a, b)$ be the unique greatest common divisor of a and b.*

For example we have $\gcd(120, 84) = 12$ and $\gcd(0, 0) = 0$. The following corollary now follows from Proposition 3.1.2.

Corollary 3.1.8. *Suppose that $a, b, q, r \in \mathbb{Z}$ and $a = qb + r$. We have $\gcd(a, b) = \gcd(b, r)$.*

The method of using repeated division with remainder, together with this corollary, to reduce the problem of calculating greatest common divisors is known as the *Euclidean Algorithm*. We saw it in action of above with 120 and 84. Here is another example where we are trying to compute $\gcd(525, 182)$. We have:

$$\begin{aligned} 525 &= 2 \cdot 182 + 161 \\ 182 &= 1 \cdot 161 + 21 \\ 161 &= 7 \cdot 21 + 14 \\ 21 &= 1 \cdot 14 + 7 \\ 14 &= 2 \cdot 7 + 0, \end{aligned}$$

so $\gcd(525, 182) = \gcd(7, 0) = 7$.

Let $a, b \in \mathbb{Z}$. Consider the set

$$\{ka + \ell b : k, \ell \in \mathbb{Z}\}.$$

This looks something like the *span* that you saw in Linear Algebra, but here we are only using integer coefficients, so we could describe this as the set of all *integer* combinations of a and b. Notice that if d is a common divisor of a and b, then $d \mid (ka + \ell b)$ for all $k, \ell \in \mathbb{Z}$ by Proposition 1.5.3, and hence d divides every element of this set. Applying this fact in the most interesting case where $d = \gcd(a, b)$ (since all other common divisors of a and b will divide $\gcd(a, b)$), we conclude that every element of $\{ka + \ell b : k, \ell \in \mathbb{Z}\}$ is a multiple of $\gcd(a, b)$. In other words, we have

$$\{ka + \ell b : k, \ell \in \mathbb{Z}\} \subseteq \{n \cdot \gcd(a, b) : n \in \mathbb{Z}\}.$$

What about the reverse containment? In particular, is $\gcd(a, b)$ always an element of $\{ka + \ell b : k, \ell \in \mathbb{Z}\}$? For example, is

$$12 \in \{k \cdot 120 + \ell \cdot 84 : k, \ell \in \mathbb{Z}\}?$$

We can attempt to play around to try to find a suitable value of k and ℓ, but there is a better way. Let's go back and look at the steps of the Euclidean Algorithm:

$$\begin{aligned} 120 &= 1 \cdot 84 + 36 \\ 84 &= 2 \cdot 36 + 12 \\ 36 &= 3 \cdot 12 + 0. \end{aligned}$$

Notice that the middle line can be manipulated to write 12 as an integer combination of 84 and 36:

$$12 = 1 \cdot 84 + (-2) \cdot 36.$$

With this in hand, we can work our way toward our goal by using the first line, which lets us write 36 as an integer combination of 120 and 84:

$$36 = 1 \cdot 120 + (-1) \cdot 84.$$

Now we can plug this expression of 36 in terms of 120 and 84 into the previous equation:

$$12 = 1 \cdot 84 + (-2) \cdot [1 \cdot 120 + (-1) \cdot 84].$$

From here, we can manipulate this equation (performing only additions and multiplications on the coefficients, not on 84 and 120 themselves!) to obtain

$$12 = (-2) \cdot 120 + 3 \cdot 84.$$

We now generalize this idea and prove that it is always possible to express $\gcd(a, b)$ as an integer combination of a and b. The proof is inductive, and follows a similar strategy to the proof of Theorem 3.1.3. Given $a, b \in \mathbb{N}$, here is the idea. To express $\gcd(a, b)$ as an integer combination of a and b, we first fix $q, r \in \mathbb{N}$ with $a = qb + r$ and $0 \leq r < b$. Now since (b, r) is "smaller" than (a, b), we inductively write $\gcd(b, r)$ as an integer combination of b and r. We then use this combination together with the equation $a = qb + r$ to write $\gcd(a, b)$ as an integer combination of a and b. Notice the similarity to the above argument where we have $120 = 1 \cdot 84 + 36$, and we used a known way to write 12 as an integer combination of 84 and 36 in order to write 12 as an integer combination of 120 and 84.

Theorem 3.1.9. *For all $a, b \in \mathbb{Z}$, there exist $k, \ell \in \mathbb{Z}$ with $\gcd(a, b) = ka + \ell b$.*

Proof. We begin by proving existence in the special case where $a, b \in \mathbb{N}$. We use induction on b to prove the result. That is, we let

$$X = \{b \in \mathbb{N} : \text{For all } a \in \mathbb{N}, \text{ there exist } k, \ell \in \mathbb{Z} \text{ with } \gcd(a, b) = ka + \ell b\}$$

and prove that $X = \mathbb{N}$ by strong induction.

- *Base Case:* Suppose that $b = 0$. Let $a \in \mathbb{N}$ be arbitrary. We then have that

  $$\gcd(a, b) = \gcd(a, 0) = a$$

  Since $a = 1 \cdot a + 0 \cdot b$, so we may let $k = 1$ and $\ell = 0$. Since $a \in \mathbb{N}$ was arbitrary, we conclude that $0 \in X$.

- *Inductive Step:* Suppose then that $b \in \mathbb{N}^+$ and we know the result for all smaller nonnegative values. In other words, we are assuming that $c \in X$ whenever $0 \leq c < b$. We prove that $b \in X$. Let $a \in \mathbb{N}$ be arbitrary. From above, we may fix $q, r \in \mathbb{Z}$ with $a = qb + r$ and $0 \leq r < b$. We also know from above that $\gcd(a, b) = \gcd(b, r)$. Since $0 \leq r < b$, we know by strong induction that $r \in X$, hence there exist $k, \ell \in \mathbb{Z}$ with

$$\gcd(b, r) = kb + \ell r$$

Now $r = a - qb$, so

$$\begin{aligned} \gcd(a, b) &= \gcd(b, r) \\ &= kb + \ell r \\ &= kb + \ell(a - qb) \\ &= kb + \ell a - qb\ell \\ &= \ell a + (k - q\ell)b. \end{aligned}$$

Since $a \in \mathbb{N}$ was arbitrary, we conclude that $b \in X$.

Therefore, we have shown that $X = \mathbb{N}$, which implies that whenever $a, b \in \mathbb{N}$, there exists $k, \ell \in \mathbb{Z}$ with $\gcd(a, b) = ka + \ell b$.

To prove the result more generally when $a, b \in \mathbb{Z}$, we again use Proposition 1.5.7. For example, if $a < 0$ but $b \geq 0$. Let $m = \gcd(a, b)$, so that $Div(m) = Div(a) \cap Div(b)$ by Proposition 3.1.5. Since $Div(-a) = Div(a)$, we also have $Div(m) = Div(-a) \cap Div(b)$, hence $m = \gcd(-a, b)$. Since $-a, b \in \mathbb{N}$, we can fix $k, \ell \in \mathbb{Z}$ with $\gcd(-a, b) = k(-a) + \ell b$. Using the fact that $\gcd(-a, b) = \gcd(a, b)$, we have $\gcd(a, b) = k(-a) + \ell b$, hence $\gcd(a, b) = (-k)a + \ell b$. Since $-k, \ell \in \mathbb{Z}$, we are done. A similar argument works if $a \geq 0$ and $b < 0$, or if both $a < 0$ and $b < 0$. □

Notice the basic structure of the above proof. If $a = qb + r$, and we happen to know $k, \ell \in \mathbb{Z}$ such that

$$\gcd(b, r) = kb + \ell r,$$

then we have

$$\gcd(a, b) = \ell a + (k - q\ell)b.$$

Given $a, b \in \mathbb{Z}$, this argument provides a recursive procedure in order to find an integer combination of a and b that gives $\gcd(a, b)$. Although the recursive procedure can be nicely translated to a computer program, we can carry it out directly by "winding up" the work created from the Euclidean Algorithm. For example, we saw above that $\gcd(525, 182) = 7$ by calculating:

$$\begin{aligned} 525 &= 2 \cdot 182 + 161 \\ 182 &= 1 \cdot 161 + 21 \\ 161 &= 7 \cdot 21 + 14 \\ 21 &= 1 \cdot 14 + 7 \\ 14 &= 2 \cdot 7 + 0. \end{aligned}$$

We now use these steps in reverse to calculate:

$$\begin{aligned}
7 &= 1 \cdot 7 + 0 \cdot 0 \\
&= 1 \cdot 7 + 0 \cdot (14 - 2 \cdot 7) \\
&= 0 \cdot 14 + 1 \cdot 7 \\
&= 0 \cdot 14 + 1 \cdot (21 - 1 \cdot 14) \\
&= 1 \cdot 21 + (-1) \cdot 14 \\
&= 1 \cdot 21 + (-1) \cdot (161 - 7 \cdot 21) \\
&= (-1) \cdot 161 + 8 \cdot 21 \\
&= (-1) \cdot 161 + 8 \cdot (182 - 1 \cdot 161) \\
&= 8 \cdot 182 + (-9) \cdot 161 \\
&= 8 \cdot 182 + (-9) \cdot (525 - 2 \cdot 182) \\
&= (-9) \cdot 525 + 26 \cdot 182.
\end{aligned}$$

This wraps everything up perfectly, but it is easier to simply start at the fifth line.

Now that we've showed that $\gcd(a, b) \in \{ka + \ell b : k, \ell \in \mathbb{Z}\}$ for all $a, b \in \mathbb{Z}$, we can now completely characterize the set of all integer combinations of a and b.

Corollary 3.1.10. *For all $a, b \in \mathbb{Z}$, we have $\{ka + \ell b : k, \ell \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) : n \in \mathbb{Z}\}$.*

Proof. Let $a, b \in \mathbb{Z}$ be arbitrary. Let $m = gcd(a, b)$. We give a double containment proof.

- $\{ka + \ell b : k, \ell \in \mathbb{Z}\} \subseteq \{nm : n \in \mathbb{Z}\}$: Let $c \in \{ka + \ell b : k, \ell \in \mathbb{Z}\}$ be arbitrary, and fix $k, \ell \in \mathbb{Z}$ with $c = ka + \ell b$. Since $m = \gcd(a, b)$, we have both $m \mid a$ and $m \mid b$. Using Proposition 1.5.3, we conclude that $m \mid c$. Therefore, we can fix $n \in \mathbb{Z}$ with $c = mn$, and hence $c \in \{nm : n \in \mathbb{Z}\}$.
- $\{nm : n \in \mathbb{Z}\} \subseteq \{ka + \ell b : k, \ell \in \mathbb{Z}\}$: Let $c \in \{nm : n \in \mathbb{Z}\}$ be arbitrary, and fix $n \in \mathbb{Z}$ with $c = nm$. Since $m = \gcd(a, b)$, we can use Theorem 3.1.9 to fix $k, \ell \in \mathbb{Z}$ with $m = ka + \ell b$. Multiplying both sides of this equation by n, we have $nm = nka + n\ell b$, so $c = (nk)a + (n\ell)b$. Since $nk, n\ell \in \mathbb{Z}$, it follows that $\{nm : n \in \mathbb{Z}\} \subseteq \{ka + \ell b : k, \ell \in \mathbb{Z}\}$.

Since we have shown both containments, it follows that $\{ka + \ell b : k, \ell \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) : n \in \mathbb{Z}\}$. □

Before moving on, we work through another proof of the existence of greatest common divisors, along with the fact that we can write $\gcd(a, b)$ as an integer combination of a and b. This proof also works because of Theorem 2.3.1, but it uses well-ordering and establishes existence without a method of computation. One may ask why we bother with another proof. One answer is that this result is so fundamental and important that two different proofs help to reinforce its value. Another reason is that each proof generalizes in different ways in more abstract settings (see Abstract Algebra).

Theorem 3.1.11. *Let $a, b \in \mathbb{Z}$ with at least one of a and b nonzero. The set*

$$\{ka + \ell b : k, \ell \in \mathbb{Z}\}$$

has positive elements, and the least positive element is a greatest common divisor of a and b. In particular, for any $a, b \in \mathbb{Z}$, there exist $k, \ell \in \mathbb{Z}$ with $\gcd(a, b) = ka + \ell b$.

Proof. Let

$$S = \{ka + \ell b : k, \ell \in \mathbb{Z}\} \cap \mathbb{N}^+.$$

We first claim that $S \neq \emptyset$. If $a > 0$, then $a = 1 \cdot a + 0 \cdot b \in S$. Similarly, if $b > 0$, then $b \in S$. If $a < 0$, then $-a > 0$ and $-a = (-1) \cdot a + 0 \cdot b \in S$. Similarly, if $b < 0$, then $-b \in S$. Since at least one of a and b is

nonzero, it follows that $S \neq \emptyset$. By the Well-Ordering property of $\mathbb{N}$, we know that S has a least element. Let $m = \min(S)$. Since $m \in S$, we may fix $k, \ell \in \mathbb{Z}$ with $m = ka + \ell b$. We claim that m is a greatest common divisor of a and b.

First, we need to check that m is a common divisor of a and b. We begin by showing that $m \mid a$. Fix $q, r \in \mathbb{Z}$ with $a = qm + r$ and $0 \leq r < m$. We want to show that $r = 0$. We have

$$\begin{aligned} r &= a - qm \\ &= a - q(ak + b\ell) \\ &= (1 - qk) \cdot a + (-q\ell) \cdot b. \end{aligned}$$

Now if $r > 0$, then we have shown that $r \in S$, which contradicts the choice of m as the least element of S. Hence, we must have $r = 0$, and so $m \mid a$.

We next show that $m \mid b$. Fix $q, r \in \mathbb{Z}$ with $b = qm + r$ and $0 \leq r < m$. We want to show that $r = 0$. We have

$$\begin{aligned} r &= b - qm \\ &= b - q(ak + b\ell) \\ &= (-qk) \cdot a + (1 - q\ell) \cdot b. \end{aligned}$$

Now if $r > 0$, then we have shown that $r \in S$, which contradicts the choice of m as the least element of S. Hence, we must have $r = 0$, and so $m \mid b$.

Finally, we need to check the last condition for m to be the greatest common divisor. Let d be a common divisor of a and b. Since $d \mid a$, $d \mid b$, and $m = ka + \ell b$, we may use Proposition 1.5.3 to conclude that $d \mid m$. □

3.2 Primes and Relatively Prime Integers

We start by defining prime numbers. We choose to only consider positive natural numbers, and we also rule out the number 1 for reasons that we will explain later.

Definition 3.2.1. *An element $p \in \mathbb{Z}$ is prime if $p > 1$ and the only positive divisors of p are 1 and p. If $n \in \mathbb{Z}$ with $n > 1$ is not prime, we say that n is* composite.

Given an integer $p > 1$, we always have both $1 \mid p$ and $p \mid p$, so p is prime exactly when $|Div^+(p)| = 2$. Notice that 2 is prime, because if $d \in \mathbb{N}^+$ is such that $d \mid 2$, then $1 \leq d \leq 2$ by Proposition 1.5.4.

Proposition 3.2.2. *If $n \in \mathbb{Z}$ and $n \notin \{1, -1\}$, then there exists a prime $p \in \mathbb{N}$ with $p \mid n$.*

Proof. First notice that 2 is prime, and that $2 \mid 0$, so the statement is true for $n = 0$. Thus, by Problem 6 on Homework 2 (that $d \mid n$ if and only if $d \mid -n$), it suffices to prove the result for $n \in \mathbb{N}$ with $n \geq 2$. We do this by strong induction.

- *Base Case:* When $n = 2$, we have that 2 is prime and $2 \mid n$ trivially.
- *Inductive Step:* Let $n \in \mathbb{N}$ with $n \geq 2$ be arbitrary such that statement is true for all natural numbers k with $2 \leq k < n$. We have two cases:
  - *Case 1:* Suppose that n is prime. We have that $n \mid n$ trivially, so in this case we can just take n.
  - *Case 2:* Suppose instead that n is not prime. By definition, we can fix $d \in \mathbb{N}$ with $d \notin \{1, n\}$ such that $d \mid n$. By Proposition 1.5.4, we have $1 \leq d \leq n$, and hence $2 \leq d < n$. By induction, we can fix a prime p with $p \mid d$. By transitivity of divisibility (Proposition 1.5.2), it follows that $p \mid n$.

Thus, in either case, we have shown that there exists a prime p with $p \mid n$.

By induction, we conclude that every $n \in \mathbb{N}$ with $n \geq 2$ is divisible by some prime, which suffices by the above discussion. □

Proposition 3.2.3. *There are infinitely many primes.*

Proof. We know that 2 is a prime, so there is at least one prime. We will take an arbitrary given finite list of primes and show that there exists a prime which is omitted. Suppose then that $p_1, p_2, \ldots, p_k$ is an arbitrary finite list of prime numbers with $k \geq 1$. We show that there exists a prime not in the list. Let

$$n = p_1 p_2 \cdots p_k + 1$$

We have $n \geq 3$, so by Proposition 3.2.2 we can fix a prime q with $q \mid n$. Suppose, for the sake of obtaining a contradiction, that $q = p_i$ for some i. We then have that $q \mid n$ and also $q \mid p_1 p_2 \cdots p_k$, so $q \mid (n - p_1 p_2 \cdots p_k)$. However, this implies that $q \mid 1$, so $|q| \leq 1$ by Proposition 1.5.4, a contradiction. Therefore $q \neq p_i$ for all i, and we have succeeded in finding a prime not in the list. □

While primality is a property of certain numbers, there is another closely related property of pairs of numbers.

Definition 3.2.4. *Two integers $a, b \in \mathbb{Z}$ are* relatively prime *if* $\gcd(a, b) = 1$.

For example, we have that 40 and 33 are relatively prime (despite neither number itself being prime), either by exhaustively checking divisors, or using the Euclidean Algorithm:

$$\begin{aligned} 40 &= 1 \cdot 33 + 7 \\ 33 &= 4 \cdot 7 + 5 \\ 7 &= 1 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

Thus, $\gcd(40, 33) = \gcd(1, 0) = 1$.

Notice that if $a \mid bc$, then it might not be the case that either $a \mid b$ or $a \mid c$. For example, we have $6 \mid 10 \cdot 9$, but $6 \nmid 10$ and $6 \mid 9$. The next result says that if $a \mid bc$, and a and b are relatively prime, then we can eliminate the b to conclude that $a \mid c$. To prove this fundamental and powerful result, we will make use of all of our hard work from the last section.

Proposition 3.2.5. *Let $a, b, c \in \mathbb{Z}$. If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.*

Proof. Since $a \mid bc$, we may fix $n \in \mathbb{Z}$ with $bc = an$. Since $\gcd(a, b) = 1$, we can use Theorem 3.1.9 to fix $k, \ell \in \mathbb{Z}$ with $ak + b\ell = 1$. Multiplying this equation through by c we conclude that $akc + b\ell c = c$, so

$$\begin{aligned} c &= akc + \ell(bc) \\ &= akc + \ell(an) \\ &= a(kc + \ell n). \end{aligned}$$

Since $kc + \ell n \in \mathbb{Z}$, it follows that $a \mid c$. □

We can quickly obtain the following consequence, which is one of the most useful facts about prime numbers.

Corollary 3.2.6. *Let $p, a, b \in \mathbb{Z}$. If $p \in \mathbb{Z}$ is prime and $p \mid ab$, then either $p \mid a$ or $p \mid b$.*

Proof. Suppose that $p \mid ab$ and $p \nmid a$. Since $\gcd(a,p)$ divides p and we know that $p \nmid a$, we have $\gcd(a,p) \neq p$. As p is prime, the only other positive divisor of p is 1, so $\gcd(a,p) = 1$. Therefore, by the Proposition 3.2.5, we conclude that $p \mid b$. □

Now that we've handled the product of two numbers, we get the following corollary about finite products by a trivial induction.

Corollary 3.2.7. *Let $p, a_1, a_2, \ldots, a_n \in \mathbb{Z}$. If p is prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some i.*

Considering the special case when all of the a_i are equal, we obtain the following.

Corollary 3.2.8. *Let $p, a \in \mathbb{Z}$ and $n \in \mathbb{N}^+$. If p is prime and $p \mid a^n$, then $p \mid a$.*

Let $a, b \in \mathbb{Z}$. We know from Theorem 3.1.9 that there exists $k, \ell \in \mathbb{Z}$ with $ka + \ell b = \gcd(a,b)$. However, be careful to note that if we find $d, k, \ell \in \mathbb{Z}$ with $ak + b\ell = d$, then it need not be the case that $d = \gcd(a,b)$. Using Corollary 3.1.10, all that we can conclude in this case is that d is a *multiple* of $\gcd(a,b)$. Nonetheless, since 1 is only a multiple of 1 and -1, if we do happen to find $k, \ell \in \mathbb{Z}$ with $ak + b\ell = 1$, then we can indeed conclude that $\gcd(a,b) = 1$. We include this equivalent condition, along with another than only looks at common prime divisors, in the following result.

Proposition 3.2.9. *Let $a, b \in \mathbb{Z}$. The following are equivalent:*

1. *$\gcd(a,b) = 1$, i.e. a and b are relatively prime.*
2. *There exist $k, \ell \in \mathbb{Z}$ with $ak + b\ell = 1$.*
3. *There is no prime $p \in \mathbb{N}$ with both $p \mid a$ and $p \mid b$.*

Proof. We prove that (1) ⇒ (2) ⇒ (3) ⇒ (1).

- (1) ⇒ (2): This follows immediately from Theorem 3.1.9.
- (2) ⇒ (3): Suppose that statement (2) is true, and fix $k, \ell \in \mathbb{Z}$ with $ak + b\ell = 1$. Let $d \in \mathbb{Z}$ be arbitrary such that both $d \mid a$ and $d \mid b$. Using Proposition 1.5.3, we can immediately conclude that $d \mid ak + b\ell$, so $d \mid 1$, and hence $|d| \leq 1$ by Proposition 1.5.4. Thus, every common divisor of a and b has absolute value less than or equal to 1. Since all primes are greater than 1 by definition, we conclude that there is no prime $p \in \mathbb{N}$ with both $p \mid a$ and $p \mid b$.
- (3) ⇒ (1): We prove the contrapositive. Suppose then that statement (1) is false, i.e. that $\gcd(a,b) \neq 1$. Since $\gcd(a,b) \geq 0$ by definition, we then have that $\gcd(a,b) \notin \{1, -1\}$. Therefore, by Proposition 3.2.2, we can fix a prime $p \in \mathbb{N}$ with $p \mid \gcd(a,b)$. Since $\gcd(a,b)$ is a common divisor of a and b, we can use Proposition 1.5.2 to conclude that p is a common divisor of a and b. Thus, we have shown that statement (3) is false.

□

Corollary 3.2.10. *Let $a, b \in \mathbb{Z}$ and let $k, \ell \in \mathbb{N}^+$. If $\gcd(a,b) = 1$, then $\gcd(a^k, b^\ell) = 1$.*

Proof. Suppose that $\gcd(a,b) = 1$. If $p \in \mathbb{Z}$ was a prime such that both $p \mid a^k$ and $p \mid b^\ell$, then we would have both $p \mid a$ and $p \mid b$ by Corollary 3.2.8, contradicting the fact that $\gcd(a,b) = 1$. Therefore, there is no common prime divisor of a^k and b^ℓ, so $\gcd(a^k, b^\ell) = 1$ by Proposition 3.2.9. □

Corollary 3.2.11. *If $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathbb{Z}$ are distinct primes, and $k_1, \ldots, k_m, \ell_1, \ldots, \ell_m \in \mathbb{N}$, then $\gcd(p_1^{k_1} \cdots p_m^{k_m}, q_1^{\ell_1} \cdots q_n^{\ell_n}) = 1$.*

Proof. Suppose that $r \in \mathbb{Z}$ is a common prime divisor of $p_1^{k_1} \cdots p_m^{k_m}$ and $q_1^{\ell_1} \cdots q_n^{\ell_n}$. By Corollary 3.2.7, we can fix an i with $r \mid p_i$ and we can fix a j with $r \mid q_j$ As p_i is prime, the only positive divisors of p_i are 1 and p_i, so since $r > 1$, we must have $r = p_i$. Similarly, as q_j is prime, we must have $r = q_j$. Therefore, $p_i = q_j$, contradicting the fact that $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathbb{Z}$ are distinct primes. Hence, there is no common prime divisor of $p_1^{k_1} \cdots p_m^{k_m}$ and $q_1^{\ell_1} \cdots q_n^{\ell_n}$, so $\gcd(p_1^{k_1} \cdots p_m^{k_m}, q_1^{\ell_1} \cdots q_n^{\ell_n}) = 1$ by Proposition 3.2.9. □

3.3 Determining the Set of Divisors

Let $a \in \mathbb{Z}\backslash\{0\}$. We know from Proposition 1.5.4 that $Div(a) \subseteq \{b \in \mathbb{Z} : |b| \leq |a|\}$, so $Div(a)$ is finite. Suppose that we want to determine $Div(a)$. Of course, we could do an exhaustive search, checking each element of $\{b \in \mathbb{Z} : |b| \leq |a|\}$ individually to determine whether it belongs to $Div(a)$. However, we can do better in several ways. First, since $0 \nmid a$ and $d \mid a$ if and only if $-d \mid a$, it suffices to determine the set $Div^+(a)$, i.e. the set of *positive* divisors of a. Now if p is prime, then $Div^+(p) = \{1, p\}$, and we are done. With a bit more work, we can determine $Div^+(a)$ whenever a is a power of a prime.

Proposition 3.3.1. *For all primes $p \in \mathbb{N}$ and all $n \in \mathbb{N}^+$, we have $Div^+(p^n) = \{p^k : 0 \leq k \leq n\}$.*

Proof. Let $p \in \mathbb{N}$ be an arbitrary prime. For this fixed prime p, we prove the statement by induction on n.

- *Base Case:* Suppose that $n = 1$. Since p is prime, we know by definition that the positive divisors of p are exactly $1 = p^0$ and $p = p^1$. Therefore, $Div^+(p^n) = \{p^0, p^1\} = \{p^k : 0 \leq k \leq 1\}$.
- *Inductive Step:* Assume that the statement is true for a given $n \in \mathbb{N}^+$, i.e. assume that $Div^+(p^n) = \{p^k : 0 \leq k \leq n\}$. First notice that for any $k \in \mathbb{N}$ with $0 \leq k \leq n+1$, we have $n + 1 - k \geq 0$ and $p^{n+1} = p^k \cdot p^{n+1-k}$, so $p^k \mid p^{n+1}$. Thus, $\{p^k : 0 \leq k \leq n+1\} \subseteq Div^+(p^{n+1})$. We now prove the reverse containment. Let $a \in Div^+(p^{n+1})$ be arbitrary. By definition, we can fix $b \in \mathbb{Z}$ with $p^{n+1} = ab$. We have two cases:
  - *Case 1:* Suppose that $p \mid a$. By definition, we can fix $c \in \mathbb{Z}$ with $a = pc$. Notice that $c \in \mathbb{N}^+$ because $a, p \in \mathbb{N}^+$. Since $p^{n+1} = ab$ and $a = pc$, we have $p^{n+1} = pcb$, so dividing both sides by p, we conclude that $p^n = cb$. Since $b \in \mathbb{Z}$, it follows that $c \mid p^n$. Therefore, $c \in Div^+(p^n)$, so by induction, we know that $c \in \{p^k : 0 \leq k \leq n\}$. Fix $\ell \in \mathbb{N}$ with $0 \leq \ell \leq n$ such that $c = p^\ell$. We then have $a = pc = pp^\ell = p^{\ell+1}$, so $a \in \{p^k : 1 \leq k \leq n+1\}$, and hence $a \in \{p^k : 0 \leq k \leq n+1\}$.
  - *Case 2:* Suppose then that $p \nmid a$. Since $\gcd(a, p)$ is a nonnegative common divisor of p and a, and the only nonnegative divisors of p are 1 and p, it follows that $\gcd(a, p) = 1$. Now we have that $a \mid p^{n+1}$, so $a \mid p \cdot p^n$. Since $\gcd(a, p) = 1$, we can use Proposition 3.2.5 to conclude that $a \mid p^n$. Therefore, $a \in Div^+(p^n)$, so by induction, we know that $a \in \{p^k : 0 \leq k \leq n\}$, and hence $a \in \{p^k : 0 \leq k \leq n+1\}$

  Thus, in either case, we have shown that $a \in \{p^k : 0 \leq k \leq n+1\}$. Since $a \in Div(p^n) \cap \mathbb{N}^+$ was arbitrary, we conclude that $Div(p^n) \cap \mathbb{N}^+ \subseteq \{p^k : 0 \leq k \leq n+1\}$, which completes the inductive step.

The result follows by induction. □

For example, since 3 is prime and $243 = 3^4$, we immediately conclude that

$$\begin{aligned} Div^+(243) &= \{3^k : 0 \leq k \leq 4\} \\ &= \{3^0, 3^1, 3^2, 3^3, 3^4\} \\ &= \{1, 3, 9, 27, 243\}. \end{aligned}$$

Although this result allows us to determine the set of divisors of a power of a prime, it does not allow us to handle numbers like $36 = 2^2 \cdot 3^2$. We can determine the positive divisors of each of 2^2 and 3^2, but it's not immediately clear how to "combine" them. More generally, suppose that we have two numbers $a_1, a_2 \in \mathbb{N}^+$. Assume that the we know the set of positive divisors of each of a_1 and a_2 individually. Now if $b_1 \mid a_1$ and $b_2 \mid a_2$, then it turns out that we must have $b_1 b_2 \mid a_1 a_2$. Moreover, *every* positive divisor of $a_1 a_2$ arises in this way.

Proposition 3.3.2. *Let $a_1, a_2 \in \mathbb{N}^+$.*

1. *If* $b_1, b_2 \in \mathbb{N}^+$ *satisfy* $b_1 \mid a_1$ *and* $b_2 \mid a_2$*, then* $b_1 b_2 \mid a_1 a_2$*.*
2. *For all* $m \in \mathbb{N}^+$ *with* $m \mid a_1 a_2$ *there exists* $b_1, b_2 \in \mathbb{N}^+$ *such that* $b_1 \mid a_1$*,* $b_2 \mid a_2$*, and* $m = b_1 b_2$*.*

Proof. 1. Let $b_1, b_2 \in \mathbb{N}^+$ be arbitrary with $b_1 \mid a_1$ with $b_2 \mid a_2$. Since $b_1 \mid a_1$, we can fix $c_1 \in \mathbb{Z}$ with $a_1 = b_1 c_1$. Since $b_2 \mid a_2$, we can fix $c_2 \in \mathbb{Z}$ with $a_2 = b_2 c_2$. We then have

$$\begin{aligned} a_1 a_2 &= b_1 c_1 \cdot b_2 c_2 \\ &= c_1 c_2 \cdot b_1 b_2. \end{aligned}$$

Since $c_1 c_2 \in \mathbb{Z}$, it follows that $b_1 b_2 \mid a_1 a_2$.

2. Let $m \in \mathbb{N}^+$ be arbitrary with $m \mid a_1 a_2$. Since $m \mid a_1 a_2$, we may fix $n \in \mathbb{Z}$ with $mn = a_1 a_2$. Let $b_1 = \gcd(m, a_1)$ and notice that $b_1 > 0$ (since $a_1 > 0$), so $b_1 \in \mathbb{N}^+$. Since b_1 is a common divisor of m and a_1, we can fix $b_2, k \in \mathbb{Z}$ with $m = b_1 b_2$ and $a_1 = b_1 k$. By Problem 5 on Homework 4, we know that $\gcd(b_2, k) = 1$. Now plugging these expressions for m and a_1 into $mn = a_1 a_2$, we see that

$$b_1 b_2 n = b_1 k a_2,$$

so dividing both sides by $b_1 > 0$, it follows that

$$b_2 n = k a_2.$$

Thus, $b_2 \mid k a_2$, so as $\gcd(b_2, k) = 1$, we can use Proposition 3.2.5 to conclude that $b_2 \mid a_2$.

□

Thus, we do obtain all of the (positive) divisors of $a_1 a_2$ by multiplying together the (positive) divisors of a_1 and a_2. However, one other natural question arises. Are the resulting divisors unique (i.e. is it impossible to obtain the same divisor in two separate ways using this process)? It turns out that uniqueness can fail. For example, if $a_1 = 6$ and $a_2 = 9$, then 18 is a divisor of $a_1 a_2 = 54$ that arises from both $18 = 6 \cdot 3$ and $18 = 2 \cdot 9$. However, if a_1 and a_2 are relatively prime, then we obtain uniqueness as well.

Proposition 3.3.3. *Let* $a_1, a_2 \in \mathbb{N}^+$ *be relatively prime integers. For all* $m \in \mathbb{N}^+$ *with* $m \mid a_1 a_2$ *there exists unique* $b_1, b_2 \in \mathbb{N}^+$ *such that* $b_1 \mid a_1$*,* $b_2 \mid a_2$*, and* $m = b_1 b_2$*.*

Proof. Let $m \in \mathbb{N}^+$ be arbitrary with $m \mid a_1 a_2$. The existence of b_1 and b_2 follow immediately from Proposition 3.3.2. We now prove uniqueness. Suppose that $b_1, b_2, c_1, c_2 \in \mathbb{N}^+$ satisfy $m = b_1 b_2 = c_1 c_2$, $b_1 \mid a_1$, $b_2 \mid a_2$, $c_1 \mid a_1$, and $c_2 \mid a_2$. We need to show that $b_1 = b_2$ and also that $c_1 = c_2$. Notice that any common divisor of b_1 and c_2 is a common divisor of a_1 and a_2 (by transitivity of divisibility), so must divide 1 because $\gcd(a_1, a_2) = 1$, and hence must be an element of $\{1, -1\}$. Thus, b_1 and c_2 are relatively prime. Similarly, b_2 and c_1 are relatively prime. Now we have

$$b_1 b_2 = m = c_1 c_2,$$

so $b_1 \mid c_1 c_2$. Since $\gcd(b_1, c_2) = 1$ from above, we can use Proposition 3.2.5 to conclude that $b_1 \mid c_1$. Similarly, we have $c_1 \mid b_1 b_2$, so as $\gcd(c_1, b_2) = 1$ from above, we conclude that $c_1 \mid b_1$. Since $b_1 \mid c_1$ and $c_1 \mid b_1$, we can Corollary 1.5.5 to deduce that $b_1 = \pm c_1$. Now $b_1, c_1 \in \mathbb{N}^+$, so we must have $b_1 = c_1$. Canceling this common term in the above displayed formula, we see that $b_2 = c_2$. This gives uniqueness. □

As an example, consider $36 = 2^2 3^2$. Using Proposition 3.3.1, we know that

$$Div^+(2^2) = \{2^0, 2^1, 2^2\} = \{1, 2, 4\}$$

and

$$Div^+(3^2) = \{3^0, 3^1, 3^2\} = \{1, 3, 9\}.$$

Since 2 and 3 are distinct primes, we can apply Corollary 3.2.11 to conclude that 2^2 and 3^2 are relatively prime. Now Proposition 3.3.2 says that we can obtain every (positive) divisor of $36 = 2^2 \cdot 3^2$ by multiplying together one (positive) divisor of 2^2 and one (positive) divisor of 3^2, and that furthermore, every way of doing this results in a different divisor of 36 by Proposition 3.3.3. Therefore,

$$\begin{aligned} Div^+(36) &= \{1 \cdot 1, 2 \cdot 1, 4 \cdot 1, 1 \cdot 3, 2 \cdot 3, 4 \cdot 3, 1 \cdot 9, 2 \cdot 9, 4 \cdot 9\} \\ &= \{1, 2, 4, 3, 6, 12, 9, 18, 36\} \\ &= \{1, 2, 3, 4, 6, 9, 12, 18, 36\}. \end{aligned}$$

We can also use these results to count the cardinality of the set of positive divisors of a given number, without having to enumerate the divisors. We first introduce a definition.

Definition 3.3.4. *Define a function $d\colon \mathbb{N}^+ \to \mathbb{N}$ by letting $d(a)$ be the number of positive divisors of a, i.e. $d(a) = |Div^+(a)|$.*

Corollary 3.3.5. *For all primes $p \in \mathbb{N}$ and all $n \in \mathbb{N}^+$, we have $d(p^n) = n + 1$.*

Proof. Immediate from Proposition 3.3.1, together with the fact that $p^k \neq p^\ell$ whenever $0 \leq k < \ell \leq n$. □

Corollary 3.3.6. *If $a_1, a_2 \in \mathbb{N}^+$ are relatively prime, then $d(a_1 a_2) = d(a_1) \cdot d(a_2)$.*

Proof. Let $m = d(a_1)$ and let $n = d(a_2)$. List the distinct elements of $Div^+(a_1)$ as $b_1, b_2, \ldots, b_m$, and list the distinct elements of $Div^+(a_2)$ as $c_1, c_2, \ldots, c_n$. By Proposition 3.3.2, we have that

$$Div^+(a_1 a_2) = \{b_i c_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\},$$

Furthermore, since a_1 and a_2 are relatively prime, the elements $b_i c_j$ are distinct (i.e. if $b_i c_j = b_k c_\ell$, then $i = j$ and $k = \ell$) by Proposition 3.3.3. Therefore, we have $d(a_1 a_2) = mn$. □

For example, we can now quickly compute that

$$\begin{aligned} d(36) &= d(2^2 \cdot 3^2) \\ &= d(2^2) \cdot d(3^2) \\ &= (2 + 1) \cdot (2 + 1) \\ &= 3 \cdot 3 \\ &= 9. \end{aligned}$$

More generally, we can use these results together with repeated applications of Corollary 3.2.11 to compute $d(a)$ whenever we have written a as a product of powers of distinct primes. For example,

$$\begin{aligned} d(720) &= d(2^4 \cdot 3^2 \cdot 5^1) \\ &= d(2^4) \cdot d(3^2 \cdot 5^1) \\ &= d(2^4) \cdot d(3^2) \cdot d(5^1) \\ &= (4 + 1) \cdot (2 + 1) \cdot (1 + 1) \\ &= 5 \cdot 3 \cdot 2 \\ &= 30. \end{aligned}$$

3.4 The Fundamental Theorem of Arithmetic

At the end of the previous section, we determined a way to compute $d(a)$, provided that we can write a as a product of powers of distinct primes. Can we always accomplish such a task? If so, is there always a unique such product? We begin by answering the first question. Note that when we say "product of primes", we are allowing the degenerate possibility of a 1-term product. That is, we still say that 2 is a product of primes, simply because 2 itself is prime.

Proposition 3.4.1. *Every $n \in \mathbb{N}$ with $n > 1$ can be written as a product of primes.*

Proof. We prove the result by strong induction on $\mathbb{N}$. If $n = 2$, we are done because 2 itself is prime. Suppose that $n > 2$ and we have proven the result for all k with $1 < k < n$. If n is prime, we are done. Suppose that n is not prime and fix a divisor $c \mid n$ with $1 < c < n$. Fix $d \in \mathbb{N}$ with $cd = n$. We then have that $1 < d < n$, so by induction, both c and d are products of primes, say $c = p_1 p_2 \cdots p_k$ and $d = q_1 q_2 \cdots q_\ell$ with each p_i and q_j prime. We then have

$$n = cd = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_\ell,$$

so n is a product of primes. The result follows by induction. □

Corollary 3.4.2. *Every $a \in \mathbb{Z}$ with $a \notin \{-1, 0, 1\}$ can be written as either a product of primes, or -1 times a product of primes.*

We now have all the tools necessary to prove the uniqueness of prime factorizations.

Theorem 3.4.3 (Fundamental Theorem of Arithmetic)**.** *Every natural number greater than 1 factors uniquely (up to order) into a product of primes. In other words, if $n \geq 2$ and*

$$p_1 p_2 \cdots p_k = n = q_1 q_2 \cdots q_\ell$$

with $p_1 \leq p_2 \leq \cdots \leq p_k$ and $q_1 \leq q_2 \leq \cdots \leq q_\ell$ all primes, then $k = \ell$ and $p_i = q_i$ for $1 \leq i \leq k$.

Proof. Existence follows from Proposition 3.4.1. We prove uniqueness by (strong) induction on n. Let $n \in \mathbb{N}$ with $n \geq 2$, and assume that every $m \in \mathbb{N}$ with $2 \leq m < n$ factors uniquely into a product of primes. We prove that n factors uniquely into primes. Let $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_\ell \in \mathbb{N}$ be primes with

$$p_1 p_2 \cdots p_k = n = q_1 q_2 \cdots q_\ell,$$

and where $p_1 \leq p_2 \leq \cdots \leq p_k$ and $q_1 \leq q_2 \leq \cdots \leq q_\ell$. We need to show that $k = \ell$ and that $p_i = q_i$ for all i. We have two cases:

- *Case 1:* Suppose that n is prime. Notice that $p_i \mid n$ for all i and $q_j \mid n$ for all j. Since the only positive divisors of n are 1 and n, and 1 is not prime, we conclude that $p_i = n$ for all i and $q_j = n$ for all j. If $k \geq 2$, then $p_1 p_2 \cdots p_k = n^k > n$, a contradiction, so we must have $k = 1$. Similarly we must have $\ell = 1$.
- *Case 2:* Suppose now that n is composite. We then must have $k \geq 2$ and $\ell \geq 2$. Now $p_1 \mid q_1 q_2 \cdots q_\ell$, so by Corollary 3.2.7, we can fix a j such that $p_1 \mid q_j$. Since q_j is prime and $p_1 \neq 1$, we must have $p_1 = q_j$. Similarly, we must have $q_1 = p_i$ for some i. We then have

  $$p_1 = q_j \geq q_1 = p_i \geq p_1,$$

  hence all inequalities must be equalities, and we conclude that $p_1 = q_1$. Canceling, we conclude that

  $$p_2 \cdots p_k = q_2 \cdots q_\ell,$$

  and this common value is some natural number m with $2 \leq m < n$. By induction, it follows that $k = \ell$ and $p_i = q_i$ for all i with $2 \leq i \leq k$.

□

Given a natural number $n \in \mathbb{N}$ with $n \geq 2$, when we write its prime factorization, we typically group together like primes and write

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where the p_i are distinct primes. We often allow the insertion of "extra" primes in the factorization of n by permitting some α_i to equal to 0. This convention is particularly useful when comparing prime factorization of two numbers so that we can assume that both factorizations have the same primes occurring. It also allows us to write 1 in such a form by choosing all α_i to equal 0. Here is one example.

Proposition 3.4.4. *Suppose that $n, d \in \mathbb{N}^+$. Write the prime factorizations of n and d as*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$
$$d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$

where the p_i are distinct primes and possibly some α_i and β_j are 0. We then have that $d \mid n$ if and only if $0 \leq \beta_i \leq \alpha_i$ for all i.

Proof. Suppose first that $0 \leq \beta_i \leq \alpha_i$ for all i. We then have that $\alpha_i - \beta_i \geq 0$ for all i, so we may let

$$c = p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2 - \beta_2} \cdots p_k^{\alpha_k - \beta_k} \in \mathbb{N}.$$

Notice that

$$\begin{aligned} dc &= p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \cdot p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2 - \beta_2} \cdots p_k^{\alpha_k - \beta_k} \\ &= (p_1^{\beta_1} p_1^{\alpha_1 - \beta_1})(p_2^{\beta_2} p_2^{\alpha_2 - \beta_2}) \cdots (p_k^{\beta_k} p_k^{\alpha_k - \beta_k}) \\ &= p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \\ &= n, \end{aligned}$$

hence $d \mid n$.

Conversely, suppose that $d \mid n$ and fix $c \in \mathbb{Z}$ with $dc = n$. Notice that $c > 0$ because $d, n > 0$. Now we have $dc = n$, so $c \mid n$. If q is prime and $q \mid c$, then $q \mid n$ by transitivity of divisibility (Proposition 1.5.2), so $q \mid p_i$ for some i by Corollary 3.2.7, and hence $q = p_i$ for some i because each p_i is prime. Thus, we can write the prime factorization of c as

$$c = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k},$$

where again we may have some γ_i equal to 0. We then have

$$\begin{aligned} n &= dc \\ &= (p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k})(p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}) \\ &= (p_1^{\beta_1} p_1^{\gamma_1})(p_2^{\beta_2} p_2^{\gamma_2}) \cdots (p_k^{\beta_k} p_k^{\gamma_k}) \\ &= p_1^{\beta_1 + \gamma_1} p_2^{\beta_2 + \gamma_2} \cdots p_k^{\beta_k + \gamma_k}. \end{aligned}$$

By the Fundamental Theorem of Arithmetic, we have $\beta_i + \gamma_i = \alpha_i$ for all i. Since $\beta_i, \gamma_i, \alpha_i \geq 0$ for all i, we conclude that $\beta_i \leq \alpha_i$ for all i. □

Corollary 3.4.5. *Let $a, b \in \mathbb{N}^+$ with and write*

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$
$$b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$

where the p_i are distinct primes. We then have

$$\gcd(a, b) = p_1^{\min\{\alpha_1, \beta_1\}} p_2^{\min\{\alpha_2, \beta_2\}} \cdots p_k^{\min\{\alpha_k, \beta_k\}}.$$

Proof. Let $m = p_1^{\min\{\alpha_1,\beta_1\}} p_2^{\min\{\alpha_2,\beta_2\}} \cdots p_k^{\min\{\alpha_k,\beta_k\}}$ and notice that $m \geq 1$ trivially. Since $\min\{\alpha_i, \beta_i\} \leq \alpha_i$ for all i, it follows from Proposition 3.4.4 that $m \mid a$. Similarly, since $\min\{\alpha_i, \beta_i\} \leq \beta_i$ for all i, it follows that $m \mid b$. Therefore, m is a common divisor of a and b.

Now let d be an arbitrary common divisor of a and b. By Proposition 3.4.4, we can write $d = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}$ with $\gamma_i \leq \alpha_i$ and $\gamma_i \leq \beta_i$ for all i. Since $\gamma_i \leq \alpha_i$ and $\gamma_i \leq \beta_i$ for all i, it follows that $\gamma_i \leq \min\{\alpha_i, \beta_i\}$ for all i. Therefore, we have both $d \mid m$ by Proposition 3.4.4.

Putting these facts together, we conclude that $m = \gcd(a, b)$. □

We can also obtain the formula for $d(n)$ that we derived in the previous section by appealing to these results.

Corollary 3.4.6. *Suppose that $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where the p_i are distinct primes. We then have*

$$d(n) = \prod_{i=1}^{k} (\alpha_i + 1).$$

Proof. Using Proposition 3.4.4, we know that a given $d \in \mathbb{N}^+$ is a divisor of n if and only if it can be written as

$$d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

where $0 \leq \beta_i \leq \alpha_i$ for all i. Notice that we have $\alpha_i + 1$ many choices for each β_i. Furthermore, different choices of β_i give rise to different values of d by the Fundamental Theorem of Arithmetic. □

Theorem 3.4.7. *Let $m, n \in \mathbb{N}$ with $m, n \geq 2$. If the unique prime factorization of m does not have the property that every prime exponent is divisible by n, then $\sqrt[n]{m}$ is irrational.*

Proof. We prove the contrapositive. Suppose that $\sqrt[n]{m}$ is rational and fix $a, b \in \mathbb{N}^+$ with $\sqrt[n]{m} = \frac{a}{b}$ (we may assume that $a, b > 0$ because $\sqrt[n]{m} > 0$). We then have

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n = m,$$

hence

$$a^n = b^n m.$$

Write a, b, m in their unique prime factorizations as

$$\begin{aligned} a &= p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \\ b &= p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \\ m &= p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}, \end{aligned}$$

where the p_i are distinct (and possibly some $\alpha_i, \beta_i, \gamma_i$ are equal to 0). Since $a^n = b^n m$, we have

$$p_1^{n\alpha_1} p_2^{n\alpha_2} \cdots p_k^{n\alpha_k} = p_1^{n\beta_1+\gamma_1} p_2^{n\beta_2+\gamma_2} \cdots p_k^{n\beta_k+\gamma_k}.$$

By the Fundamental Theorem of Arithmetic, we conclude that $n\alpha_i = n\beta_i + \gamma_i$ for all i. Therefore, for each i, we have $\gamma_i = n\alpha_i - n\beta_i = n(\alpha_i - \beta_i)$, and so $n \mid \gamma_i$ for each i. □

Chapter 4

Injections, Surjections, and Bijections

4.1 Definitions and Examples

Recall that the defining property of a function $f\colon A \to B$ is that every input element from A produces a unique output element from B. However, this does not work in reverse. Given $b \in B$, it may be the case that b is the output of zero, one, or many elements from A. We give special names to the types of functions where we have limitations for how often elements $b \in B$ actually occur as an output.

Definition 4.1.1. *Let $f\colon A \to B$ be a function.*

- *We say that f is* injective *(or* one-to-one*) if whenever $a_1, a_2 \in A$ satisfy $f(a_1) = f(a_2)$, we have $a_1 = a_2$.*
- *We say that f is* surjective *(or* onto*) if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.*
- *We say that f is* bijective *if f is both injective and surjective.*

Let's take a moment to unpack these definitions. First, saying that function $f\colon A \to B$ is surjective is simply saying that every $b \in B$ is hit at least once by an element $a \in A$. We can rephrase this using Definition 1.4.3 by saying that $f\colon A \to B$ is surjective exactly when $\text{range}(f) = B$.

The definition of injective is slightly more mysterious at first. Intuitively, a function $f\colon A \to B$ is injective if every $b \in B$ is hit by at most one $a \in A$. Now saying this precisely takes a little bit of thought. After all, how can we say "there exists at most one" because our "there exists" quantifier is used to mean that there is at least one! The idea is to turn this around and not directly talk about $b \in B$ at all. Instead, we want to say that we never have a situation where we have two distinct elements $a_1, a_2 \in A$ that go to the same place under f. Thus, we want to say

"**Not** (There exists $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$)".

We can rewrite this statement as

"For all $a_1, a_2 \in A$, we have **Not**($a_1 \neq a_2$ and $f(a_1) = f(a_2)$)",

which is equivalent to

"For all $a_1, a_2 \in A$, we have either $a_1 = a_2$ or $f(a_1) \neq f(a_2)$"

(notice that the negation of the "and" statement turned into an "or" statement). Finally, we can rewrite this as the following "if...then..." statement:

"For all $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$".

Looking at our statement here, it captures what we want to express perfectly because it says that distinct inputs always go to distinct outputs, which exactly says no element of B is hit by 2 or more elements, and hence that every element of B is hit by at most 1 element. Thus, we could indeed take this as our definition of injective. The problem is that this definition is difficult to use in practice. To see why, think about how we would argue that a given function $f\colon A \to B$ is injective. It appears that we would want to take arbitrary $a_1, a_2 \in A$ with $a_1 \neq a_2$, and argue that under this assumption we must have that $f(a_1) \neq f(a_2)$. Now the problem with this is that is very difficult to work with an expression involving $\neq$ in ways that preserve truth. For example, we have that $-1 \neq 1$, but $(-1)^2 = 1^2$, so we can not square both sides and preserve non-equality. To get around this problem, we instead take the contrapositive of the statement in question, which turns into our formal definition of injective:

"For all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$".

Notice that in our definition above, we simply replace the "for all... if... then..." construct with a "whenever...we have..." for clarity, but these are saying precisely the same thing, i.e. that whenever we have two elements of A that happen to be sent to the same element of B, then in fact those two elements of A must be the same. Although our official definition is slightly harder to wrap one's mind around, it is *much* easier to work with in practice. To prove that a given $f\colon A \to B$ is injective, we take arbitrary $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$, and use this equality to derive the conclusion that $a_1 = a_2$.

To recap the colloquial ways to understand these concepts, a function $f\colon A \to B$ is injective if every $b \in B$ is hit by at most one $a \in A$, and is surjective if every $b \in B$ is hit by at least one $a \in A$. It follows that a function $f\colon A \to B$ is bijective if every $b \in B$ is hit by exactly one $a \in A$. These ways of thinking about injective and surjective are great, but we need to be careful when proving that a function is injective or surjective. Given a function $f\colon A \to B$, here is the general process for proving that it has one or both of these properties:

- In order to prove that f is injective, you should start by taking arbitrary $a_1, a_2 \in A$ that satisfy $f(a_1) = f(a_2)$, and then work forward to derive that $a_1 = a_2$. In this way, you show that whenever two elements of A happen to go to the same output, then they must have been the same element all along.

- In order to prove that f is surjective, you should start by taking an arbitrary $b \in B$, and then show how to build an $a \in A$ with $f(a) = b$. In other words, you want to take an arbitrary $b \in B$ and fill in the blank in $f(\_\_\_\_) = b$ with an element of A.

Here is an example.

Proposition 4.1.2. *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is bijective.*

Proof. We need to show that f is both injective and surjective.

- We first show that f is injective. Let $x_1, x_2 \in \mathbb{R}$ be arbitrary with $f(x_1) = f(x_2)$. We then have that $2x_1 = 2x_2$. Dividing both sides by 2, we conclude that $x_1 = x_2$. Since $x_1, x_2 \in \mathbb{R}$ were arbitrary with $f(x_1) = f(x_2)$, it follows that f is injective.

- We next show that f is surjective. Let $y \in \mathbb{R}$ be arbitrary. Notice that $\frac{y}{2} \in \mathbb{R}$ and that

$$f\left(\frac{y}{2}\right) = 2 \cdot \frac{y}{2} = y.$$

  Thus, we have shown the existence of an $x \in \mathbb{R}$ with $f(x) = y$. Since $y \in \mathbb{R}$ was arbitrary, it follows that f is surjective

Since f is both injective and surjective, it follows that f is bijective. □

Notice that if we define $g\colon \mathbb{Z} \to \mathbb{Z}$ by letting $g(x) = 2x$, then g is injective by the same proof, but g is not surjective because there does not exist $m \in \mathbb{Z}$ with $f(m) = 1$ (since this would imply that $2m = 1$, so $2 \mid 1$, a contradiction). Thus, changing the domain or codomain of a function can change the properties of that function.

Proposition 4.1.3. *The function $d\colon \mathbb{N}^+ \to \mathbb{N}^+$, where $d(n)$ is the number of positive divisors of n, is surjective but not injective.*

Proof. Both 3 and 5 are prime, so $d(3) = 2 = d(5)$. Since $3 \neq 5$, it follows that d is not injective. To show that d is surjective, first notice that $d(1) = 1$, so $1 \in \text{range}(d)$. Now given an arbitrary $m \in \mathbb{N}^+$ with $m \geq 2$, we have that $m - 1 \in \mathbb{N}^+$, so

$$d(2^{m-1}) = (m-1) + 1 = m$$

by Corollary 3.3.5 (since 2 is prime). Therefore, $\text{range}(d) = \mathbb{N}^+$, and hence d is surjective. □

Here are several more examples, where $|\sigma|$ is defined to be the length of the sequence σ:

- $f\colon \{0,1\}^* \to \mathbb{N}$ defined by $f(\sigma) = |\sigma|$ is surjective but not injective.
- $f\colon \{0,1\}^* \to \mathbb{Z}$ defined by $f(\sigma) = |\sigma|$ is neither surjective nor injective.
- $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is neither injective nor surjective.

Proposition 4.1.4. *Let A, B, C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions*

1. *If f and g are both injective, then $g \circ f$ is injective.*
2. *If f and g are both surjective, then $g \circ f$ is surjective.*
3. *If f and g are both bijective, then $g \circ f$ is bijective.*
4. *If $g \circ f$ is injective, then f is injective.*
5. *If $g \circ f$ is surjective, then g is surjective.*

Proof.

1. Suppose that f and g are both injective. Let $a_1, a_2 \in A$ be arbitrary with $(g \circ f)(a_1) = (g \circ f)(a_2)$. By definition of composition, we then have $g(f(a_1)) = g(f(a_2))$. Using the fact that g is injective, we conclude that $f(a_1) = f(a_2)$. Now we use the fact that f is injective to conclude that $a_1 = a_2$. Therefore, $g \circ f$ is injective.

2. Suppose that f and g are both surjective. Let $c \in C$ be arbitrary. Since g is surjective, we can fix $b \in B$ with $g(b) = c$. Since f is surjective, we can fix $a \in A$ with $f(a) = b$. We then have

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= g(b) \\ &= c\end{aligned}$$

   Since $c \in C$ was arbitrary, we conclude that $g \circ f$ is surjective.

3. This follows from combining 1 and 2.

4. Suppose that $g \circ f$ is injective. Let $a_1, a_2 \in A$ be arbitrary with $f(a_1) = f(a_2)$. Applying g to both sides, we then have that $g(f(a_1)) = g(f(a_2))$, so $(g \circ f)(a_1) = (g \circ f)(a_2)$. Using the fact that $g \circ f$ is injective, it follows that $a_1 = a_2$. Therefore, f is injective.

5. Suppose that $g \circ f$ is surjective. Let $c \in C$ be arbitrary. Since $g \circ f$ is surjective, we can fix $a \in A$ with $(g \circ f)(a) = c$. By definition of composition, we then have $g(f(a)) = c$. Since $f(a) \in B$, we have succeeded in finding a b with $g(b) = c$ (namely $b = f(a)$). Since $c \in C$ was arbitrary, we conclude that g is surjective.

□

Definition 4.1.5. *Let A be a set. The function $id_A \colon A \to A$ defined by $id_A(a) = a$ for all $a \in A$ is called the* identity function *on A.*

We call this function the identity function because it leaves other functions alone when we compose with it. However, we have to be careful that we compose with the identity function on the correct set and the correct side.

Proposition 4.1.6. *For any function $f \colon A \to B$, we have $f \circ id_A = f$ and $id_B \circ f = f$.*

Proof. Let $f \colon A \to B$ be an arbitrary function.

- We first show that $f \circ id_A = f$. Let $a \in A$ be arbitrary. We have

$$\begin{aligned} (f \circ id_A)(a) &= f(id_A(a)) && \text{(by definition of composition)} \\ &= f(a) \end{aligned}$$

  Since $a \in A$ was arbitrary, it follows that $f \circ id_A = f$.

- We now show that $id_B \circ f = f$. Let $a \in A$ be arbitrary. We have

$$\begin{aligned} (id_B \circ f)(a) &= id_B(f(a)) && \text{(by definition of composition)} \\ &= f(a) && \text{(because } f(a) \in B\text{)} \end{aligned}$$

  Since $a \in A$ was arbitrary, it follows that $id_B \circ f = f$.

□

Suppose that $f \colon A \to B$ is a function. We want to think about what an *inverse function* of f would even mean. Naturally, an inverse should "undo" what f does. Since $f \colon A \to B$, we should think about functions $g \colon B \to A$. In order for g to undo f, it seems that we would want $g(f(a)) = a$ for all $a \in A$. Similarly, we might want this to work in the other direction so that $f(g(b)) = b$ for all $b \in B$. Notice that we can write the statement "$g(f(a)) = a$ for all $a \in A$" in a more elegant fashion by saying that $g \circ f = id_A$, where $id_A \colon A \to A$ is the identity function on A (i.e. $id_A(a) = a$ for all $a \in A$). Similarly, we can write "$f(g(b)) = b$ for all $b \in B$" as $f \circ g = id_B$. We codify these ideas in a definition.

Definition 4.1.7. *Let $f \colon A \to B$ be a function.*

- *A* left inverse *for f is a function $g \colon B \to A$ such that $g \circ f = id_A$.*
- *A* right inverse *for f is a function $g \colon B \to A$ such that $f \circ g = id_B$.*
- *An* inverse *for f is a function $g \colon B \to A$ such that both $g \circ f = id_A$ and $f \circ g = id_B$.*

Let's consider an example. Let $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$, and consider the function $f \colon A \to B$ defined as follows:

$$f(1) = 7 \qquad f(2) = 5 \qquad f(3) = 8.$$

As a set, we can write $f = \{(1, 7), (2, 5), (3, 8)\}$. Notice that f is injective but not surjective because $6 \notin \text{range}(f)$. Does f have a left inverse or a right inverse? A guess would be to define $g \colon B \to A$ as follows:

$$g(5) = 2 \qquad g(6) = ? \qquad g(7) = 1 \qquad g(8) = 3.$$

Notice that it is unclear how to define $g(6)$ because 6 is not hit by f. Suppose that we pick a random $c \in A$ and let $g(6) = c$. We have the following:

$$\begin{array}{rcccl} g(f(1)) & = & g(7) & = & 1 \\ g(f(2)) & = & g(5) & = & 2 \\ g(f(3)) & = & g(8) & = & 3. \end{array}$$

Thus, we have $g(f(a)) = a$ for all $a \in A$, so $g \circ f = id_A$ regardless of how we choose to define $g(6)$. We have shown that f has a left inverse (in fact, we have shown that f has at least 3 left inverses because we have 3 choices for $g(6)$). Notice that the value of $g(6)$ never came up in the above calculation because $6 \notin \text{range}(f)$. What happens when we look at $f \circ g$? Ignoring 6 for the moment, we have the following:

$$\begin{array}{rcccl} f(g(5)) & = & f(2) & = & 5 \\ f(g(7)) & = & f(1) & = & 7 \\ f(g(8)) & = & f(3) & = & 8. \end{array}$$

Thus, we have $f(g(b)) = b$ for all $b \in \{5, 7, 8\}$. However, notice that no matter how we choose $c \in A$ to define $g(6) = c$, it doesn't work. For example, if let $g(6) = 1$, then $f(g(6)) = f(1) = 7$. You can work through the other two possibilities directly, but notice that no matter how we choose c, we will have $f(g(c)) \in \text{range}(f)$, and hence $f(g(c)) \neq 6$ because $6 \notin \text{range}(f)$. In other words, it appears that f does not have a right inverse. Furthermore, this problem seems to arise whenever we have a function that is not surjective.

Let's see an example where f is not injective. Let $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, and consider the function $f\colon A \to B$ defined as follows:

$$f(1) = 5 \qquad f(2) = 6 \qquad f(3) = 5.$$

As a set, we can write $f = \{(1, 5), (2, 6), (3, 5)\}$. Notice that f is surjective but not injective (since $f(1) = f(3)$ but $1 \neq 3$). Does f have a left inverse or a right inverse? The guess would be to define $g\colon B \to A$ by letting $g(6) = 2$, but it's unclear how to define $g(5)$. Should we let $g(5) = 1$ or should we let $g(5) = 3$? Suppose that we choose $g\colon B \to A$ as follows:

$$g(5) = 1 \qquad g(6) = 2.$$

Let's first look at $f \circ g$. We have the following:

$$\begin{array}{rcccl} f(g(5)) & = & f(1) & = & 5 \\ f(g(6)) & = & f(2) & = & 6. \end{array}$$

We have shown that $f(g(b)) = b$ for all $b \in B$, so $f \circ g = id_B$. Now if we instead choose $g(5) = 3$, then we would have

$$\begin{array}{rcccl} f(g(5)) & = & f(3) & = & 5 \\ f(g(6)) & = & f(2) & = & 6, \end{array}$$

which also works. Thus, we have shown that f has a right inverse, and in fact it has at least 2 right inverses. What happens if we look at $g \circ f$ for these functions g? If we define $g(5) = 1$, then we have

$$g(f(3)) = g(5) = 1,$$

which does not work. Alternatively, if we define $g(5) = 3$, then we have

$$g(f(1)) = g(5) = 3,$$

which does not work either. It seems that no matter how we choose $g(5)$, we will obtain the wrong result on some input to $g \circ f$. In other words, it appears that f does not have a left inverse. Furthermore, this problem seems to arise whenever we have a function that is not injective.

Now if $f\colon A \to B$ is bijective, then it seems reasonable that if we define $g\colon B \to A$ by simply "flipping all of the arrows", then g will be an inverse for f (on both sides), and that this is the only possible way to define an inverse for f. We now prove all of these results in general, although feel free to skim the next couple of results for now (we won't need them for a little while).

Proposition 4.1.8. *Let $f\colon A \to B$ be a function.*

1. *f is injective if and only if there exists a left inverse for f.*
2. *f is surjective if and only if there exists a right inverse for f.*
3. *f is bijective if and only if there exists an inverse for f.*

Proof.

1. Suppose first that f has a left inverse, and fix a function $g\colon B \to A$ with $g \circ f = id_A$. Suppose that $a_1, a_2 \in A$ satisfy $f(a_1) = f(a_2)$. Applying the function g to both sides we see that $g(f(a_1)) = g(f(a_2))$, and hence $(g \circ f)(a_1) = (g \circ f)(a_2)$. We now have

$$\begin{aligned} a_1 &= id_A(a_1) \\ &= (g \circ f)(a_1) \\ &= (g \circ f)(a_2) \\ &= id_A(a_2) \\ &= a_2 \end{aligned}$$

   so $a_1 = a_2$. It follows that f is injective.

   Suppose conversely that f is injective. If $A = \emptyset$, then $f = \emptyset$, and we are done by letting $g = \emptyset$ (if the empty set as a function annoys you, just ignore this case). Let's assume then that $A \neq \emptyset$ and fix $a_0 \in A$. We now define $g\colon B \to A$. Given $b \in B$, we define $g(b)$ as follows:

   - If $b \in \text{range}(f)$, then there exists a unique $a \in A$ with $f(a) = b$ (because f is injective), and we let $g(b) = a$ for this unique choice.
   - If $b \notin \text{range}(f)$, then we let $g(b) = a_0$.

   This completes the definition of $g\colon B \to A$. In terms of sets, g is obtained from f by flipping all of the pairs, and adding (b, a_0) for all $b \notin \text{range}(f)$. We need to check that $g \circ f = id_A$. Let $a \in A$ be arbitrary. We then have that $f(a) \in B$, and furthermore $f(a) \in \text{range}(f)$ trivially. Therefore, in the definition of g on the input $f(a)$, we defined $g(f(a)) = a$, so $(g \circ f)(a) = id_A(a)$. Since $a \in A$ was arbitrary, it follows that $g \circ f = id_A$. Therefore, f has a left inverse.

2. Suppose first that f has a right inverse, and fix a function $g\colon B \to A$ with $f \circ g = id_B$. Let $b \in B$ be arbitrary. We then have that

$$\begin{aligned} b &= id_B(b) \\ &= (f \circ g)(b) \\ &= f(g(b)) \end{aligned}$$

   hence there exists $a \in A$ with $f(a) = b$, namely $a = g(b)$. Since $b \in B$ was arbitrary, it follows that f is surjective.

Suppose conversely that f is surjective. We define $g\colon B \to A$ as follows. For every $b \in B$, we know that there exists (possibly many) $a \in A$ with $f(a) = b$ because f is surjective. Given $b \in B$, we then define $g(b) = a$ for some (any) $a \in A$ for which $f(a) = b$. Now given any $b \in B$, notice that $g(b)$ satisfies $f(g(b)) = b$ by definition of g, so $(f \circ g)(b) = b = id_B(b)$. Since $b \in B$ was arbitrary, it follows that $f \circ g = id_B$.

3. The right to left direction is immediate from parts 1 and 2. For the left to right direction, we need only note that if f is a bijection, then the function g defined in the left to right direction in the proof of 1 equals the function g defined in the left to right direction in the proof of 2.

□

Proposition 4.1.9. *Let $f\colon A \to B$ be a function. If $g\colon B \to A$ is a left inverse of f and $h\colon B \to A$ is a right inverse of f, then $g = h$.*

Proof. By definition, we have that that $g \circ f = id_A$ and $f \circ h = id_B$. The key function to consider is the composition $(g \circ f) \circ h = g \circ (f \circ h)$ (notice that these are equal by Proposition 1.4.5). We have

$$\begin{aligned} g &= g \circ id_B \\ &= g \circ (f \circ h) \\ &= (g \circ f) \circ h && \text{(by Proposition 1.4.5)} \\ &= id_A \circ h \\ &= h. \end{aligned}$$

Therefore, we conclude that $g = h$. □

Corollary 4.1.10. *If $f\colon A \to B$ is a function, then there exists at most one function $g\colon B \to A$ that is an inverse of f.*

Proof. Suppose that $g\colon B \to A$ and $h\colon B \to A$ are both inverses of f. In particular, we then have that g is a left inverse of f and h is a right inverse of f. Therefore, $g = h$ by Proposition 4.1.9. □

Corollary 4.1.11. *If $f\colon A \to B$ is a bijective function, then there exists a unique inverse for f.*

Proof. Immediate from Proposition 4.1.8 and Corollary 4.1.10. □

Notation 4.1.12. *Suppose that $f\colon A \to B$ is bijective. We let $f^{-1}\colon B \to A$ be the the unique inverse for f. More concretely, f^{-1} is defined as follows. Given $b \in B$, we define $f^{-1}(b)$ to equal the unique $a \in A$ with $f(a) = b$.*

Notice that by definition, we have both $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$. In other words, we have $f^{-1}(f(a)) = a$ for all $a \in A$, and $f(f^{-1}(b)) = b$ for all $b \in B$.

4.2 The Bijection Principle

Perhaps somewhat surprisingly, we can use functions to help us determine the cardinality of a set. The following fact connects up the concepts introduced in the previous section with cardinalities of the domain and codomain of functions.

Fact 4.2.1. *Let A and B be finite sets.*

- *There exists an injective function $f\colon A \to B$ if and only if $|A| \leq |B|$.*

- *There exists a surjective function* $f\colon A \to B$ *if and only if* $|B| \leq |A|$.
- *There exists a bijective function* $f\colon A \to B$ *if and only if* $|A| = |B|$.

It is reasonably straightforward to provide intuitive arguments for each of these facts. Let's look at the first one. Suppose first that there exists an injective function $f\colon A \to B$. In this case, every element of B is hit by at most one element of A via f, so there must be at least as many elements in B as there are in A. For the converse, if $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ with $m \leq n$, then we can define an injective function $f\colon A \to B$ by letting $f(a_i) = b_i$ for all i. Although each of these arguments is convincing, the first one is not terribly precise. If desired, it is possible to prove the first one by induction on the cardinalities of A and B, but just as for the Sum Rule we will avoid being so formal. The other two results can be argued similarly.

The third fact listed above is very helpful when trying to determine the cardinality of a set, and is sometimes called the "Bijection Principle". Perhaps surprisingly, we've already used this type of argument informally on several occasions. On the first homework, we showed that if A is a set with $|A| = n$ and $D = \{(a, a) : a \in A\}$, then we had $|D| = n$. Intuitively, if $A = \{a_1, a_2, \ldots, a_n\}$, then $D = \{(a_1, a_1), (a_2, a_2), \ldots, (a_n, a_n)\}$, so $|D| = n$. Using the Bijection Principle, we can argue this more formally by exhibiting the following function: define $f\colon A \to D$ by letting $f(a) = (a, a)$. It is then straightforward to check that f is bijective, and hence $|D| = |A| = n$.

We were also implicitly using the bijection principle in the proof of Corollary 3.3.6, which said that $d(a_1a_2) = d(a_1) \cdot d(a_2)$ whenever $a_1, a_2 \in \mathbb{N}^+$ were relatively prime. The idea behind the argument was that in this case, every (positive) divisor of a_1a_2 can be decomposed uniquely as a product of a (positive) divisor of a_1 and a (positive) divisor of a_2. More formally, we can use the first part of Proposition 3.3.2 to define the function

$$f\colon Div^+(a_1) \times Div^+(a_2) \to Div^+(a_1a_2)$$

given by letting $f(d_1, d_2) = d_1d_2$. Notice that f is surjective by the second part of Proposition 3.3.2, and f is injective by Proposition 3.3.3 (which is the only place where we use that a_1 and a_2 are relatively prime). Therefore, by the Bijection Principle, we know that

$$|Div^+(a_1) \times Div^+(a_2)| = |Div^+(a_1a_2)|.$$

Now the Product Rule tells us that

$$|Div^+(a_1) \times Div^+(a_2)| = |Div^+(a_1)| \cdot |Div^+(a_2)|,$$

so

$$|Div^+(a_1)| \cdot |Div^+(a_2)| = |Div^+(a_1a_2)|,$$

and hence $d(a_1) \cdot d(a_2) = d(a_1a_2)$.

Finally, we can see the Bijection Principle at work in the proof of Corollary 3.4.6. Suppose that $n > 1$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ where the p_i are distinct primes. Consider the set

$$S = \{0, 1, \ldots, \alpha_1\} \times \{0, 1, \ldots, \alpha_2\} \times \cdots \times \{0, 1, \ldots, \alpha_k\}.$$

Since $|\{0, 1, \ldots, \alpha_k\}| = \alpha_i + 1$ for all i, we can use the General Product Rule to conclude that

$$|S| = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1) = \prod_{i=1}^{k}(\alpha_i + 1).$$

Define $f\colon S \to \mathbb{N}^+$ by letting $f(\beta_1, \beta_2, \ldots, \beta_k) = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$. The Fundamental Theorem of Arithmetic tells us that f is injective, and Proposition 3.4.4 allows us to conclude that range$(f) = Div^+(n)$. Thus, if

we restrict the codomain to view $f\colon S \to Div^+(n)$, then f is bijective. Using the Bijection Principle, we conclude that $|S| = |Div^+(n)|$, so

$$d(n) = |S| = \prod_{i=1}^{k}(\alpha_i + 1).$$

In general, if we have a set A and want to determine $|A|$, the idea is to create a bijection between A and another set B, where $|B|$ is easier to determine. Our first new example of this technique is the following fundamental result:

Proposition 4.2.2. *Given a finite set A with $|A| = n \in \mathbb{N}^+$, there exists a bijection $f\colon \{0,1\}^n \to \mathcal{P}(A)$.*

Before jumping into the proof, we first illustrate with a special case. Let $A = \{1,2,3\}$, so that $|A| = 3$. Notice that

$$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Looking at the elements of $\mathcal{P}(\{1,2,3\})$, we see that to create a subset of $\{1,2,3\}$, we need to decide which elements to keep, and which elements to omit. In other words, when building a subset of $\{1,2,3\}$, we ask ourselves whether to include 1 in our subset, whether to include 2 in our subset, and whether to include 3 in our subset. Each of these is a binary choice, and can be coded by either a 0 or a 1. Given a 3-tuple of 0's and 1's, say $(1,0,1)$, we can think the first 1 as coding the information that we should include 1 in our set, the 0 as coding that we should omit 2, and the 1 as coding that we should include 3. Thus, we associate $(1,0,1)$ with the subset $\{1,3\}$ of $\{1,2,3\}$. In this way, we establish the following bijection between $\{0,1\}^3$ and $\mathcal{P}(\{1,2,3\})$:

$$\begin{aligned}
(0,0,0) &\mapsto \emptyset \\
(0,0,1) &\mapsto \{3\} \\
(0,1,0) &\mapsto \{2\} \\
(1,0,0) &\mapsto \{1\} \\
(0,1,1) &\mapsto \{2,3\} \\
(1,0,1) &\mapsto \{1,3\} \\
(1,1,0) &\mapsto \{1,2\} \\
(1,1,1) &\mapsto \{1,2,3\}.
\end{aligned}$$

We now write the general proof.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ where the a_i are distinct. Define a function $f\colon \{0,1\}^n \to \mathcal{P}(A)$ by letting $f(b_1, b_2, \ldots, b_n) = \{a_i : b_i = 1\}$. In other words, given a finite sequence $(b_1, b_2, \ldots, b_n)$ of 0's and 1's, we send it to the subset of A obtained by including a_i precisely when the i^{th} element of the sequence is a 1. Notice that if $(b_1, b_2, \ldots, b_n) \neq (c_1, c_2, \ldots, c_n)$, then we can fix an i with $b_i \neq c_i$, and in this case we have $f(b_1, b_2, \ldots, b_n) \neq f(c_1, c_2, \ldots, c_n)$ because a_i is one of the sets but not the other. Furthermore, given any $S \subseteq A$, if we let $(b_1, b_2, \ldots, b_n) \in \{0,1\}^n$ be defined by letting

$$b_i = \begin{cases} 1 & \text{if } a_i \in S \\ 0 & \text{if } a_i \notin S, \end{cases}$$

then $f(b_1, b_2, \ldots, b_n) = S$, so f is surjective. Therefore, f is a bijection. □

Corollary 4.2.3. *If $|A| = n \in \mathbb{N}^+$, then $|\mathcal{P}(A)| = 2^n$.*

Proof. This is immediate from the Proposition 4.2.2, the Bijection Principle, and Corollary 1.2.11. □

Since this result is so fundamental, we give another proof that uses both induction and the Bijection Principle. As above, we first provide some intuition by considering $\mathcal{P}(\{1,2,3\})$. Notice that we can break up $\mathcal{P}(\{1,2,3\})$ into the union of two disjoint subsets, consisting of those elements that do not contain 3, and those that do contain 3:

$$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \cup \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Notice that first of these subsets is just $\mathcal{P}(\{1,2\})$, and the second can be created from the first by inserting 3 into each of the subsets. In other words, we can use induction to determine the cardinality of the first set, and then notice that there is a bijection between the subsets that do not contain 3, and those that do contain 3. Here is the general argument:

Proof 2 of Corollary 4.2.3. We prove the result by induction on $n \in \mathbb{N}^+$.

- *Base Case:* Suppose that $n = 1$. Let A be a set with $|A| = 1$, say $A = \{a\}$. We then have that $\mathcal{P}(A) = \{\emptyset, \{a\}\}$, so $|\mathcal{P}(A)| = 2 = 2^1$.
- *Induction Step:* Assume that the statement is true for some fixed $n \in \mathbb{N}^+$, i.e. assume that for some fixed $n \in \mathbb{N}^+$, we know that $|\mathcal{P}(A)| = 2^n$ for all sets A with $|A| = n$. Consider an arbitrary set A with $|A| = n+1$. Fix some (any) element $a_0 \in A$. Let $\mathcal{S} \subseteq \mathcal{P}(A)$ be the collection of subsets of A not having a_0 as an element, and let $\mathcal{T} \subseteq \mathcal{P}(A)$ be the collection of subsets of A having a_0 as an element. Notice then that $\mathcal{S}$ and $\mathcal{T}$ are disjoint sets with $\mathcal{P}(A) = \mathcal{S} \cup \mathcal{T}$, so by the Sum Rule we know that

  $$|\mathcal{P}(A)| = |\mathcal{S}| + |\mathcal{T}|.$$

  Now consider the function $f\colon \mathcal{S} \to \mathcal{T}$ defined by letting $f(B) = B \cup \{a_0\}$, i.e. given $B \in \mathcal{S}$, we have that B is a subset of A not having a_0 as an element, and we send it to the subset of A obtained by throwing a_0 in as a new element. Notice that f is a bijection, so $|\mathcal{S}| = |\mathcal{T}|$. Therefore, we have

  $$|\mathcal{P}(A)| = |\mathcal{S}| + |\mathcal{S}|.$$

  Finally, notice that $\mathcal{S} = \mathcal{P}(A \backslash \{a_0\})$, so since $|A \backslash \{a_0\}| = n$, we can use induction to conclude that $|A \backslash \{a_0\}| = 2^n$. Therefore,

  $$\begin{aligned} |\mathcal{P}(A)| &= 2^n + 2^n \\ &= 2 \cdot 2^n \\ &= 2^{n+1}. \end{aligned}$$

  Thus, the statement is true for $n+1$.

By induction, we conclude that if $|A| = n \in \mathbb{N}^+$, then $|\mathcal{P}(A)| = 2^n$. □

By the way, Corollary 4.2.3 is also true in the case $n = 0$. When $n = 0$, we have $A = \emptyset$ and $\mathcal{P}(\emptyset) = \{\emptyset\}$, so $|\mathcal{P}(\emptyset)| = 1 = 2^0$.

We can also use the Bijection Principle when we do not know the size of either set. For an illustrative example, consider the set $\mathcal{P}(\{1,2,3,4,5\})$. We know from Corollary 4.2.3 that $|\mathcal{P}(\{1,2,3,4,5\})| = 2^5 = 32$. What if we only wanted to consider the subsets of $\{1,2,3,4,5\}$ that have exactly 2 elements? If we let $\mathcal{S}$ be this subset of $\mathcal{P}(\{1,2,3,4,5\})$, then

$$\mathcal{S} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{3,4\}\}.$$

Similarly, if we let $\mathcal{T}$ be the subsets of $\{1,2,3,4,5\}$ having exactly 3 elements, then

$$\mathcal{T} = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}\}.$$

In this case, you can directly check that $|\mathcal{S}| = 10 = |\mathcal{T}|$, but we can also argue that $|\mathcal{S}| = |\mathcal{T}|$ without performing an exhaustive count. The key idea is that the relative complement of a size 2 subset of $\{1, 2, 3, 4, 5\}$ is a size 3 subset of $\{1, 2, 3, 4, 5\}$. In other words, we have the following bijection between $\mathcal{S}$ and $\mathcal{T}$:

$$\begin{aligned}
\{1, 2\} &\mapsto \{3, 4, 5\} \\
\{1, 3\} &\mapsto \{2, 4, 5\} \\
\{1, 4\} &\mapsto \{2, 3, 5\} \\
\{1, 5\} &\mapsto \{2, 3, 4\} \\
\{2, 3\} &\mapsto \{1, 4, 5\} \\
\{2, 4\} &\mapsto \{1, 3, 5\} \\
\{2, 5\} &\mapsto \{1, 3, 4\} \\
\{3, 4\} &\mapsto \{1, 2, 5\} \\
\{3, 5\} &\mapsto \{1, 2, 4\} \\
\{4, 5\} &\mapsto \{1, 2, 3\}.
\end{aligned}$$

Generalizing this argument leads to following result.

Proposition 4.2.4. *Let A be a set with $|A| = n \in \mathbb{N}^+$ and let $k \in \mathbb{N}$ be such that $0 \leq k \leq n$. The number of subsets of A having cardinality k equals the number of subsets of A having cardinality $n - k$.*

Proof. Let $\mathcal{S}$ be the collection of all subsets of A having cardinality k, and let $\mathcal{T}$ be the collection of all subsets of A having cardinality $n - k$. Define $f \colon \mathcal{S} \to \mathcal{T}$ by letting $f(B) = A \backslash B$, i.e. given $B \subseteq A$ with $|B| = k$, send it to the complement of B in A (notice that if $|B| = k$, then $|A \backslash B| = n - k$ by the complement rule). Notice that f is a bijection (it is surjective because if $C \subseteq A$ is such that $|C| = n - k$, then $|A \backslash C| = k$ and $f(A \backslash C) = C$). Therefore, $|\mathcal{S}| = |\mathcal{T}|$. □

Thus, despite the fact that we do not (yet) have a formula for the number of subsets of a certain size, we do know that the number of subsets of size k must equal the number of subsets of size $n - k$.

4.3 The Pigeonhole Principle

Although we have thus far focused on bijections, we also know from the previous section that if A and B are finite sets, and $f \colon A \to B$ is an injective function, then $|A| \leq |B|$. Taking the contrapositive of this fact, we obtain the following:

Corollary 4.3.1 (Pigeonhole Principle)**.** *If A and B are finite sets with $|A| > |B|$, and $f \colon A \to B$ is a function, then there exist $a_1, a_2 \in A$ with $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$.*

Stated informally, the Pigeonhole Principle says that if $n > k$ and we place n balls into k boxes, then (at least) one box will contain at least 2 balls. For a very simple example, in any group of 13 people, there must exist (at least) 2 people in the group who were born in the same month. Here is a more interesting example:

Proposition 4.3.2. *Given $n + 1$ integers, it is always possible to find two whose difference is divisible by n.*

Proof. Let A be a set of $n+1$ integers, so $A = \{a_0, a_1, \ldots, a_n\}$. For each i, we can use division with remainder to fix $q_i, r_i \in \mathbb{Z}$ with

$$a_i = nq_i + r_i$$

and $0 \leq r_i < n$. Notice then that $r_i \in \{0, 1, 2, \ldots, n - 1\}$ for all i. Define $f \colon A \to \{0, 1, 2, \ldots, n - 1\}$ by letting $f(a_i) = r_i$ for each i. Since $|A| = n + 1$ and $|\{0, 1, 2, \ldots, n - 1\}| = n$, we know by the Pigeonhole

Principle that f is not injective. Thus, we can fix $i \neq j$ with $r_i = r_j$. We then have

$$\begin{aligned} a_i - a_j &= (nq_i + r_i) - (nq_j + r_j) \\ &= n(q_i - q_j) + (r_i - r_j) \\ &= n(q_i - q_j) && (\text{since } r_i - r_j = 0), \end{aligned}$$

so $n \mid (a_i - a_j)$. □

Proposition 4.3.3. *For each $n \in \mathbb{N}^+$, let $a_n = 333\cdots3$ where there are n many 3s. There exists an $n \leq 1492$ such that $1491 \mid a_n$.*

Proof. For each n with $1 \leq n \leq 1492$, we can use division with remainder to fix $q_i, r_i \in \mathbb{Z}$ with

$$a_i = 1491q_i + r_i$$

and $0 \leq r_i < 1491$. Since we have 1491 many possible distinct r_i, it follows that there exists $i < j$ with $r_i = r_j$. We then have

$$1491 \mid (a_j - a_i).$$

as in the proof of the previous proposition. The problem is that $a_j - a_i$ does not equal any of the a_n. However, notice that

$$a_j - a_i = 333\cdots300\cdots0 = a_{j-i} \cdot 10^i,$$

so as $1491 \mid (a_j - a_i)$, it follows that

$$1491 \mid a_{j-i} \cdot 10^i.$$

Now the prime factorization of 10^i is 2^i5^i, so the any prime divisor of 10^i must divide either 2 or 5 by Proposition 3.2.7, so must be either 2 or 5 because 2 and 5 are prime. Now we have $1491 = 2 \cdot 745 + 1$ and $1491 = 5 \cdot 298 + 1$, so $2 \nmid 1491$ and $5 \nmid 1491$ by Proposition 2.3.5. Using Proposition 3.2.9, it follows that $\gcd(1491, 10^i) = 1$. As a result, we can apply Proposition 3.2.5 to conclude that $1491 \mid a_{j-i}$, completing the proof. □

Notice that the above argument works if we replace 1491 by any number that ends in 1, 3, 7, or 9 (since such a number is not divisible by 2 or 5), and we if replace the 3 in $333\cdots3$ by any nonzero digit.

Proposition 4.3.4. *Suppose we have a gathering of $n \geq 2$ people, and at the beginning of the gathering some pairs of people shake hands. There always must exist (at least) two people who have shaken the same number of hands.*

Proof. Label the people with the numbers $1, 2, 3, \ldots, n$. We can then define a function $f\colon \{1, 2, 3, \ldots, n\} \to \{0, 1, 2, \ldots, n-1\}$ by letting $f(k)$ be the number of people that person k shook hands with. On the face of it, this looks bad because both sets have n elements. However, it is impossible that both 0 and $n-1$ are elements of range(f) because if somebody shook hands with all of the other $n-1$ people, then everybody shook hands with a least one person, so $0 \notin \text{range}(f)$. Thus, we can either view f as a function $f\colon \{1, 2, 3, \ldots, n\} \to \{0, 1, 2, \ldots, n-2\}$ or as a function $f\colon \{1, 2, 3, \ldots, n\} \to \{1, 2, \ldots, n-1\}$. In either case, f is not injective by the Pigeonhole Principle, so there exist two people who have shaken the same number of hands. □

Proposition 4.3.5. *Let $f\colon \{0,1\}^* \to \{0,1\}^*$ be injective. For every $n \in \mathbb{N}^+$, there exists $\sigma \in \{0,1\}^n$ with $|f(\sigma)| \geq |\sigma|$ (here $|\tau|$ is the length of the finite sequence τ).*

Proof. Let $f\colon \{0,1\}^* \to \{0,1\}^*$ be injective. Let $n \in \mathbb{N}^+$ be arbitrary. Suppose instead that $|f(\sigma)| < |\sigma|$ for all $\sigma \in \{0,1\}^n$. Notice that $|\{0,1\}|^n = 2^n$ and the number of sequences of length strictly less than n is $1 + 2 + 4 + \cdots + 2^{n-1}$ because we can write it as the union $\{0,1\}^0 \cup \{0,1\}^1 \cup \{0,1\}^2 \cup \cdots \cup \{0,1\}^{n-1}$ where the sets are pairwise disjoint. Now the key fact is that

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

by the homework. Alternatively, this can be argued by noting that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^{n-1} &= (1 + 2 + 2^2 + \cdots + 2^{n-1}) \cdot 1 \\ &= (1 + 2 + 2^2 + \cdots + 2^{n-1}) \cdot (2 - 1) \\ &= (2 + 2^2 + 2^3 + \cdots + 2^n) - (1 + 2 + 2^2 + \cdots + 2^{n-1}) \\ &= 2^n - 1. \end{aligned}$$

Since $|\{0,1\}^n| = 2^n$ and the set of sequences of length strictly less than n is $2^n - 1$, we may use the Pigeonhole Principle to conclude that there exists distinct $\sigma_1, \sigma_2 \in \{0,1\}^n$ with $f(\sigma_1) = f(\sigma_2)$, which contradicts the fact that f is injective. Therefore, there must exist $\sigma \in \{0,1\}^n$ with $|f(\sigma)| \geq |\sigma|$. □

We can interpret the previous proposition as follows. Suppose that we have a compression algorithm, i.e. a program that takes a sequence of 0's and 1's and tries to compress it down to a shorter sequence (think of any standard zip program). If we look at how the function behaves on every input, we obtain a function $f\colon \{0,1\}^* \to \{0,1\}^*$. Of course, for this compression algorithm to be at all useful, we would need to be able to uncompress any file back to its original. In order to do this, the function f must be injective (otherwise, if two files compress to the same thing, we would have no way to know which file to return). This proposition says that any purported compression scheme must fail to actually shrink the size of some file, and in fact for *every* length n, there is a file of length n that is not actually made smaller.

Proposition 4.3.6. *Let $n \in \mathbb{N}^+$. Given a set $S \subseteq \{1, 2, 3, \ldots, 2n\}$ with $|S| \geq n + 1$, there always exists a pair of distinct elements $a, b \in S$ with $a \mid b$.*

Before proving this proposition, we examine some special cases in order to get some intuition. First, consider the case when $n = 5$ so that $2n = 10$. We want to prove that whenever we have at least 6 numbers from the set $\{1, 2, 3, \ldots, 10\}$, we can find two distinct numbers a and b such that $a \mid b$. The idea is to build five "boxes" of numbers with the following properties:

- Every number from $\{1, 2, 3, \ldots, 10\}$ is in a box.
- Given any two distinct numbers from the same box, one divides the other.

Suppose that we are successful in doing this. Then given any set of at least six numbers, we can find two of the numbers in the same box (because we only five boxes), and then we will be done. So let's build five boxes with the above properties in the case where $n = 5$:

- Box 1: $\{1, 2, 4, 8\}$.
- Box 2: $\{3, 6\}$.
- Box 3: $\{5, 10\}$.
- Box 4: $\{7\}$.
- Box 5: $\{9\}$.

We now want to generalize this argument. The key idea behind the above boxes was as follows: Given a natural number, keep dividing by 2 until we reach an odd number, and put two numbers in the same box if we arrive at the same odd number. In order to formalize this, we prove the following lemma.

Lemma 4.3.7. *Let $n \in \mathbb{N}^+$. There exist unique $k, \ell \in \mathbb{N}$ such that ℓ is odd and $n = 2^k \ell$.*

Although it is possible to deduce this result from the Fundamental Theorem of Arithmetic, we give a direct proof.

Proof. We first prove the existence of k and ℓ by strong induction on n.

- When $n = 1$, we can write $1 = 2^0 \cdot 1$, so we can take $k = 0$ and $\ell = 1$.
- Let $n \in \mathbb{N}$ with $n \geq 2$, and assume that we know the existence part is true for all m with $1 \leq m < n$. We prove it for n. First, notice that if n is odd, then we can simply write $n = 2^0 n$, and we are done. Suppose then that n is even. Fix $m \in \mathbb{Z}$ with $n = 2m$ and notice that $1 \leq m < n$. By induction, we can fix $k, \ell \in \mathbb{N}$ such that ℓ is odd and $m = 2^k \ell$. We then have $n = 2m = 2^{k+1}\ell$, hence the result holds for n.

The existence of k and ℓ for all n follows by induction.

We now prove uniqueness. Suppose that $k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ are such that ℓ_1 and ℓ_2 are both odd and $2^{k_1}\ell_1 = 2^{k_2}\ell_2$. If $k_1 < k_2$, then dividing both sides by 2^{k_1}, we would be able to conclude that $\ell_1 = 2^{k_2 - k_1}\ell_2$, which contradicts the fact that ℓ_1 is odd (since $k_2 - k_1 \geq 1$). A similar contradiction occurs if $k_1 > k_2$. Therefore, we must have that $k_1 = k_2$. Diving both sides by $2^{k_1} = 2^{k_2}$, we then conclude that $\ell_1 = \ell_2$. This gives uniqueness. □

Proof of Proposition 4.3.6. Let $S \subseteq \{1, 2, 3, \ldots, 2n\}$ with $|S| \geq n + 1$ be arbitrary. Let X be the set of all odd integers ℓ with $1 \leq \ell \leq 2n$, and notice that $|X| = n$ (because $g \colon \{0, 1, 2, \ldots, n-1\} \to X$ given by $g(k) = 2k + 1$ is a bijection). Define a function $f \colon S \to X$ as follows. Given $a \in S$, write $a = 2^k \ell$ for the unique k and ℓ from the previous lemma, and define $f(a) = \ell$ (notice that $\ell \leq 2n$ because $a \leq 2n$). Intuitively, we associate to each given $n \in S$ the unique odd number obtained by repeatedly dividing by 2 until we reach an odd number. Since $|S| \geq n + 1$ and $|X| = n$, the Pigeonhole Principle tells us that we can find distinct $a, b \in S$ with $a < b$ such that $f(a) = f(b)$. Call this common value ℓ, i.e. let $\ell = f(a) = f(b)$, and fix $k_1, k_2 \in \mathbb{N}$ with $a = 2^{k_1}\ell$ and $b = 2^{k_2}\ell$. Since $a < b$, we have $k_1 < k_2$. Now

$$\begin{aligned} b &= 2^{k_2}\ell \\ &= 2^{k_2 - k_1} \cdot 2^{k_1} \cdot \ell \\ &= 2^{k_2 - k_1} \cdot a, \end{aligned}$$

so $a \mid b$. This completes the proof. □

We end with a more sophisticated example. Suppose that we have a finite sequence of (possibly real) numbers. For example, consider the following sequence of 10 numbers:

$$3 \quad 1 \quad 6 \quad 9 \quad 0 \quad 2 \quad 8 \quad 5 \quad 7 \quad 4$$

Although these numbers are not sorted in any sense, one can find a decently long decreasing subsequence by pulling out the $9, 8, 7, 4$. It turns out that no matter what sequence of length n one looks at, it always possible to pull out an increasing or decreasing subsequence of length about $\sqrt{n}$. We first provide the necessary definitions:

Definition 4.3.8. *Suppose that $a_1, a_2, \ldots, a_n$ is a finite sequence of real numbers. Suppose that we have a sequence of indices with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We then call $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ a* subsequence *of $a_1, a_2, \ldots, a_n$.*

Definition 4.3.9. *Suppose that $a_1, a_2, \ldots, a_n$ is a finite sequence of real numbers.*

- *We call the sequence* increasing *if $a_1 \leq a_2 \leq \cdots \leq a_n$.*
- *We call the sequence* decreasing *if $a_1 \geq a_2 \geq \cdots \geq a_n$.*
- *We call the sequence* monotonic *if it is either increasing or decreasing.*

For example, suppose that $a_1, a_2, \ldots, a_{10}$ is our original sequence

$$3 \quad 1 \quad 6 \quad 9 \quad 0 \quad 2 \quad 8 \quad 5 \quad 7 \quad 4$$

Notice that $9, 8, 7, 4$ is a decreasing subsequence of this sequence (with $i_1 = 4$, $i_2 = 7$, $i_3 = 9$, and $i_4 = 10$).

Theorem 4.3.10. *Let $n \in \mathbb{N}^+$. Given a sequence of $(n-1)^2 + 1$ real numbers, it is always possible to find a monotonic subsequence of length n.*

Proof. Consider an arbitrary sequence

$$a_1, a_2, a_3, \ldots, a_{(n-1)^2+1}$$

of $(n-1)^2+1$ many real numbers. Associate to each i the pair $(k, \ell) \in \mathbb{N}^+ \times \mathbb{N}^+$, where k is the length of the longest increasing subsequence ending with (and including) a_i and ℓ is the length of the longest decreasing subsequence ending with (and including) a_i. If any one of these pairs has a coordinate that is at least n, then we are done. Otherwise, every pair (k, ℓ) is such that $1 \leq k \leq n-1$ and $1 \leq \ell \leq n-1$. There are only $(n-1)^2$ many possible pairs, so since we have $(n-1)^2 + 1$ many numbers, some pair must be repeated by the Pigeonhole Principle. Fix $i < j$ with $(k_i, \ell_i) = (k_j, \ell_j)$. Now if $a_j \geq a_i$, then we can add a_j onto the end of the longest increasing subsequence ending in a_i to form an increasing subsequence of length $k_i + 1 > k_j$, a contradiction. Similarly, if $a_j \leq a_i$, then we can add a_j onto the end of the longest decreasing subsequence ending in a_i to form an idecreasing subsequence of length $\ell_i + 1 > \ell_j$, a contradiction. □

For example, for our sequence

$$3 \quad 1 \quad 6 \quad 9 \quad 0 \quad 2 \quad 8 \quad 5 \quad 7 \quad 4$$

we would assign the values

$$(1,1) \quad (1,2) \quad (2,1) \quad (3,1) \quad (1,3) \quad (2,2) \quad (3,2) \quad (3,3) \quad (4,3) \quad (3,4)$$

Thus, we can take either $0, 2, 5, 7$ as an increasing subsequence or $9, 8, 7, 4$ as a decreasing subsequence.

4.4 Countability and Uncountability

Recall that if A and B are finite sets, then $|A| = |B|$ if and only if there exists a bijection $f \colon A \to B$. Now if A and B are infinite sets, then we have no obvious way to define the cardinality of A and B like we do for finite sets. However, it still makes sense to talk about bijections, and so one can simply *define* two (possibly infinite) sets A and B to have the same size if there is a bijection $f \colon A \to B$.

With this in mind, think about $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and the subset $\mathbb{N}^+ = \{1, 2, 3, 4, \ldots\}$. Although $\mathbb{N}^+$ is a proper subset of $\mathbb{N}$ and "obviously" has one fewer element, the function $f \colon \mathbb{N} \to \mathbb{N}^+$ given by $f(n) = n + 1$ is a bijection, and so $\mathbb{N}$ and $\mathbb{N}^+$ have the same "size". For another even more surprising example, let $A = \{2n : n \in \mathbb{N}\} = \{0, 2, 4, 6, \ldots\}$ be the set even natural numbers, and notice that the function $f \colon \mathbb{N} \to A$ given by $f(n) = 2n$ is a bijection from $\mathbb{N}$ to A. Hence, even though A intuitively seems to only have "half" of the elements of $\mathbb{N}$, there is still a bijection between $\mathbb{N}$ and A.

The next proposition shows that $\mathbb{N}$ is the "smallest" infinite set, in the sense that we can injectively embed it into any infinite set.

Proposition 4.4.1. *If A is an infinite set, then there is an injective function $f\colon \mathbb{N} \to A$.*

Proof. We define $f\colon \mathbb{N} \to A$ recursively. Pick some (any) $a_0 \in A$, and define $f(0) = a_0$. Suppose that $n \in \mathbb{N}$ and we have defined the values $f(0), f(1), \ldots, f(n)$, all of which are elements of A. Since A is infinite, we have that $\{f(0), f(1), \ldots, f(n)\} \neq A$. Thus, we can pick some $a_{n+1} \in A \backslash \{f(0), f(1), \ldots, f(n)\}$, and define $f(n+1) = a_{n+1}$. With this recursive definition, we have defined a function $f\colon \mathbb{N} \to A$. Notice that if $m < n$, then $f(n)$ was chosen to be distinct from $f(m)$ by definition, so $f(m) \neq f(n)$. Therefore, f is injective. □

With this in mind, we introduce a name for those infinite sets for which we can find a bijection with $\mathbb{N}$, and think of them as the "smallest" types of infinite sets.

Definition 4.4.2. *Let A be a set.*

- *We say that A is* countably infinite *if there exists a bijection $f\colon \mathbb{N} \to A$.*
- *We say that A is* countable *if it is either finite or countably infinite.*
- *If A is not countable, we say that A is* uncountable.

Suppose that A is countably infinite. We then have a bijection $f\colon \mathbb{N} \to A$, so we can arrange its elements in a list without repetition by listing out $f(0), f(1), f(2), f(3), \ldots$ to get:

$$a_0 \quad a_1 \quad a_2 \quad a_3 \quad \cdots$$

Conversely, writing out such a list without repetition shows how to build a bijection $f\colon \mathbb{N} \to A$. Since working with such lists is more intuitively natural (although perhaps a little less rigorous), we'll work with countable sets in this way. What about lists that allow repetitions?

Proposition 4.4.3. *Let A be a set. The following are equivalent.*

1. *It is possible to list A, possibly with repetition, as $a_0, a_1, a_2, a_3, \ldots$.*
2. *There is a surjection $g\colon \mathbb{N} \to A$.*
3. *A is countable, i.e. either finite or countably infinite.*

Proof. (1) ⇔ (2): This is essentially the same as the argument just given. If we can list A, possibly with repetitions, as $a_0, a_1, a_2, a_3, \ldots$, then the function $g\colon \mathbb{N} \to A$ given by $g(n) = a_n$ is a surjection. Conversely, if there is a surjection $g\colon \mathbb{N} \to A$, then $g(0), g(1), g(2), g(3), \ldots$ is a listing of A.

(1) ⇒ (3): Assume (1), and fix a listing $a_0, a_1, a_2, a_3, \ldots$ of A, possibly with repetition. If A is finite, then A is countable by definition, so we may assume that A is infinite. We define a new list as follows. Let $b_0 = a_0$. If we have defined $b_0, b_1, \ldots, b_n$, let $b_{n+1} = a_k$, where k is chosen as the least value such that $a_k \notin \{b_0, b_1, \ldots, b_n\}$ (such a k exists because A is infinite). Then

$$b_0 \quad b_1 \quad b_2 \quad b_3 \quad \cdots$$

is a listing of A without repetitions. Therefore, A is countably infinite.

(3) ⇒ (1): Suppose that A is countable. If A is countably infinite, then there is a bijection $f\colon \mathbb{N} \to A$, in which case

$$f(0) \quad f(1) \quad f(2) \quad f(3) \quad \cdots$$

is a listing of A (even without repetition). On the other hand, if A is finite, say $A = \{a_0, a_1, a_2, \ldots, a_n\}$, then

$$a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_n \quad a_n \quad a_n \quad \cdots$$

is a listing of A (with repetition). □

Our first really interesting result is that $\mathbb{Z}$, the set of integers, is countable. Of course, some insight is required because if we simply start to list the integers as

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots$$

we won't ever get to the negative numbers. We thus use the sneaky strategy of bouncing back-and-forth between positive and negative integers.

Proposition 4.4.4. *$\mathbb{Z}$ is countable.*

Proof. We can list $\mathbb{Z}$ as

$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad \cdots$$

More formally, we could define $f\colon \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

and check that f is a bijection. □

The key idea used in previous proof can be abstracted into the following result.

Proposition 4.4.5. *If A and B are countable, then $A \cup B$ is countable.*

Proof. Since A is countable, we may list it as $a_0, a_1, a_2, a_3, \ldots$. Since B is countable, we may list it as $b_0, b_1, b_2, b_3, \ldots$. We therefore have the following two lists:

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & \cdots \\ b_0 & b_1 & b_2 & b_3 & \cdots \end{array}$$

We can list $A \cup B$ by going back-and-forth between the above lists as

$$a_0 \quad b_0 \quad a_1 \quad b_1 \quad a_2 \quad b_2 \quad \cdots$$

□

A slightly stronger result is now immediate.

Corollary 4.4.6. *If $A_0, A_1, \ldots, A_n$ are countable, then $A_0 \cup A_1 \cup \cdots \cup A_n$ is countable.*

Proof. This follows from Proposition 4.4.5 by induction. Alternatively, we can argue as follows. For each fixed k with $0 \leq k \leq n$, we know that A_k is countable, so we may list it as $a_{k,0}, a_{k,1}, a_{k,2}, \ldots$. We can visualize the situation with the following table.

$$\begin{array}{ccccc} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & \cdots \end{array}$$

We now list $A_0 \cup A_1 \cup \cdots \cup A_n$ by moving down each column in order, to obtain:

$$a_{0,0} \quad a_{1,0} \quad \cdots \quad a_{n,0} \quad a_{0,1} \quad a_{1,1} \quad \cdots \quad a_{n,1} \quad \cdots \quad \cdots$$

□

In fact, we can prove quite a significant extension of the above results. The next proposition is usually referred to by saying that "the countable union of countable sets is countable".

Proposition 4.4.7. *If $A_0, A_1, A_2, \ldots$ are all countable, then $\bigcup_{k=0}^{\infty} A_k = A_0 \cup A_1 \cup A_2 \cup \cdots$ is countable.*

Proof. For each $n \in \mathbb{N}$, we know that A_n is countable, so we may list it as $a_{k,0}, a_{k,1}, a_{k,2}, a_{k,3}, \ldots$. We now have the following table.

$$\begin{array}{ccccc} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Now we can't list this by blindly walking down the rows or columns. We thus need a new, much more clever, strategy. The idea is to list the elements of the table by moving between rows and columns. One nice approach which works is to step along certain diagonals and obtain the following listing of $\bigcup_{n=0}^{\infty} A_n$:

$$a_{0,0} \quad a_{0,1} \quad a_{1,0} \quad a_{0,2} \quad a_{1,1} \quad a_{2,0} \quad \cdots$$

The pattern here is that we are walking along the diagonals in turn, each of which is finite. Alternatively, we can describe this list as follows. For each $m \in \mathbb{N}$, there are only finitely many pairs $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $i + j = m$. We first list the finitely many $a_{i,j}$ with $i + j = 0$, followed by those finitely many $a_{i,j}$ with $i + j = 1$, then those finitely many $a_{i,j}$ with $i + j = 2$, etc. This gives a listing of $\bigcup_{k=0}^{\infty} A_k$. □

Theorem 4.4.8. *$\mathbb{Q}$ is countable.*

Proof. For each $k \in \mathbb{N}^+$, let $A_k = \{\frac{a}{k} : a \in \mathbb{Z}\}$. Notice that each A_k is countable because we can list it as

$$\frac{0}{k} \quad \frac{1}{k} \quad \frac{-1}{k} \quad \frac{2}{k} \quad \frac{-2}{k} \quad \cdots$$

Since

$$\mathbb{Q} = \bigcup_{k=1}^{\infty} A_k = A_1 \cup A_2 \cup A_3 \cup \cdots$$

we can use Proposition 4.4.7 to conclude that $\mathbb{Q}$ is countable. □

With all of this in hand, it is natural to ask whether uncountable sets exist.

Theorem 4.4.9. *$\mathbb{R}$ is uncountable.*

Proof. We need to show that there is no list of real numbers that includes every element of $\mathbb{R}$. Suppose then that $r_1, r_2, r_3, \ldots$ is an arbitrary list of real numbers. We show that there exists $x \in \mathbb{R}$ with $x \neq r_n$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we write out the (nonterminating) decimal expansion of r_n as

$$a_n \quad . \quad d_{n,1} \quad d_{n,2} \quad d_{n,3} \quad d_{n,4} \quad \cdots$$

where $a_n \in \mathbb{Z}$ and each $d_{n,i} \in \mathbb{Z}$ satisfies $0 \leq d_{n,i} \leq 9$. We arrange our list of reals $r_1, r_2, r_3, \ldots$ as a table

$$\begin{array}{ccccccc} a_1 & . & d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & \cdots \\ a_2 & . & d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & \cdots \\ a_3 & . & d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & \cdots \\ a_4 & . & d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

For each $n \in \mathbb{N}$, let

$$e_n = \begin{cases} 3 & \text{if } d_{n,n} \neq 3 \\ 7 & \text{if } d_{n,n} = 3 \end{cases}$$

Let x be the real number with decimal expansion

$$. \quad e_1 \quad e_2 \quad e_3 \quad e_4 \quad \cdots$$

We claim that $x \neq r_n$ for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary. Since $e_n \neq d_{n,n}$ by construction, it follows x and r_n disagree in the n^{th} decimal position. Therefore, since the (nonterminating) decimal expansions of x and r_n are different, it follows that $x \neq r_n$. □

Chapter 5

Counting

5.1 Arrangements, Permutations, and Combinations

Let A be a finite set with $|A| = n$. Given $k \in \mathbb{N}^+$, the set A^k is the set of all finite sequences of length k whose elements are all from A. Occasionally, and especially in computer science, such a finite sequence is called a *string* over A of length k. Using Corollary 1.2.11, we already know that $|A^k| = |A|^k = n^k$, so we can count the number of finite sequences of length k. For example, if $A = \{a, b, c, d\}$, then there are exactly $4^2 = 16$ many two letter strings over A. Similarly, there are exactly $128^2 = 16,384$ many two character long ASCII sequences, and there are 10^7 many potential phone numbers.

Recall that a finite sequence might contain repetition. For example, if $A = \{1, 2, 3\}$, then $(1, 1, 3) \in A^3$ and $(3, 1, 2, 3) \in A^4$. Suppose that A is a set with $|A| = n$, and we want to count the number of sequences of length 2 where there is no repetition, i.e. we want to determine the cardinality of the set

$$B = \{(a, b) \in A^2 : a \neq b\}.$$

There are (at least) two straightforward ways to do this:

- *Method 1:* As on the first homework, we use the complement rule. Let $D = \{(a, a) : a \in A\}$ and notice that $|D| = n$ because $|A| = n$ (and $f : A \to D$ given by $f(a) = (a, a)$ is a bijection). Since $B = A^2 \backslash D$, it follows that $|B| = |A^2| - |D| = n^2 - n = n(n - 1)$.

- *Method 2:* We use a modified version of the product rule as follows. Think about constructing an element of B in two stages. First, we need to pick the first coordinate of our pair, and we have n choices here. Now once we fix the first coordinate of our pair, we have $n - 1$ choices for the second coordinate because we can choose any element of A other than the one that we chose in the first round. By making these two choices in succession, we determine an element of B, and furthermore, every element of B is obtained via a unique sequence of such choices. Therefore, we have $|B| = n(n - 1)$.

Notice that in the argument for Method 2 above, we are not directly using the Product Rule. The issue is that we can not write B in the form $B = X \times Y$ where $|X| = n$ and $|Y| = n - 1$ because the choice of second coordinates depends upon the choice of first component. For example, if $A = \{1, 2, 3\}$, then if we choose 1 as our first coordinate, then we can choose any element of $\{2, 3\}$ for the second, while if we choose 3 as our first coordinate, then we can choose any element of $\{1, 2\}$ for the second. However, the key fact is that the *number* of choices for the second coordinate is the same no matter what we choose for the first. If you want to be more formal in the example with $A = \{1, 2, 3\}$, we are setting up a bijection between $\{1, 2, 3\} \times \{1, 2\}$

and B as follows:

$$(1,1) \mapsto (1,2)$$
$$(1,2) \mapsto (1,3)$$
$$(2,1) \mapsto (2,1)$$
$$(2,2) \mapsto (2,3)$$
$$(3,1) \mapsto (3,1)$$
$$(3,2) \mapsto (3,2)$$

In other words, if we input (k, ℓ), then the output will have first coordinate k, but the second coordinate will be the ℓ^{th} smallest element of $\{1,2,3\}\backslash\{k\}$. Moreover, if $A = \{1,2,3,\ldots,n\}$, then the function from $\{1,2,\ldots,n\} \times \{1,2,\ldots,n-1\}$ sending (k,ℓ) to the pair whose first coordinate is k, and whose second coordinate is the ℓ^{th} smallest element of $\{1,2,\ldots,n\}\backslash\{k\}$, is a bijection. Therefore, $|B| = n(n-1)$.

Suppose more generally that we are building a set of objects in stages, in such a way that each *sequence* of choices throughout the stages determines a unique object, and no two distinct sequences determine the same object. Suppose also that we have the following number of choices at each stage:

- There are n_1 many choices at the first stage.
- For each choice in the first stage, there are n_2 many objects to pair with it in the second stage.
- For each pair of choices in the first two stages, there are n_3 many choices to append at the third stage.
- ...
- For each sequence of choices in the first $k-1$ stages, there are n_k many choices to append at the k^{th} stage.

In this situation, there are $n_1 n_2 n_3 \cdots n_k$ many total objects in the set. The formal argument is similar to the above argument, where we set up a bijection from the set

$$\{1,2,\ldots,n_1\} \times \{1,2,\ldots,n_2\} \times \cdots \times \{1,2,\ldots,n_k\}$$

to the set that we are counting. However, we will avoid such formalities in the future when we use it. With this new rule in hand, we can count a new type of object.

Definition 5.1.1. *Let A be a finite set with $|A| = n$. A* permutation *of A is an element of A^n without repeated elements.*

For example, consider $A = \{1,2,3\}$. One example of a permutation of A is $(3,1,2)$. The set of all permutations of A is:

$$\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

Thus, there are 6 permutations of the set $\{1,2,3\}$. In order to count the number of permutations of a set with n elements, we use our new technique.

Proposition 5.1.2. *If A is a finite set with $n \in \mathbb{N}^+$ elements, then there are $n!$ many permutations of A.*

Proof. We can build a permutation of A through a sequence of choices.

- We being by choosing the first element, and we have n choices.
- Once we've chosen the first element, we have $n-1$ choices for the second because we can choose any element of A other than the one chosen in the first stage.

- Next, we have $n-2$ many choices for the third element.
- $\ldots$
- At stage $n-1$, we have chosen $n-2$ distinct elements so far, so we have 2 choices here.
- Finally, we have only one choice remaining for the last position.

Since every such sequence of choices determines a permutation of A, and distinct choices given distinct permutations, it follows that there are $n(n-1)(n-2)\cdots 2\cdot 1 = n!$ many permutations of A. □

Alternatively, we can give a recursive description of the number of permutations of a set with n-elements, and use that to derive the above result. Define $f\colon \mathbb{N}^+ \to \mathbb{N}^+$ by letting $f(n)$ be the number of permutations of $\{1,2,\ldots,n\}$. Notice that $f(1)=1$. Suppose that we know the value of $f(n)$. We show how to build all permutations of $\{1,2,\ldots,n+1\}$ from the $f(n)$ many permutations of $\{1,2,\ldots,n\}$ along with an element of the set $\{1,2,\ldots,n,n+1\}$. Given a permutation of $\{1,2,\ldots,n\}$ together with a number k with $1\leq k\leq n+1$, we form a permutation of $\{1,2,\ldots,n,n+1\}$ by taking our permutation of $\{1,2,\ldots,n\}$, and inserting $n+1$ into the sequence in position k (and then shifting all later numbers to the right). For example, when $n=4$ and we have the permutation $(4,1,3,2)$ together with $k=2$, then we insert 5 into the second position to form the permutation $(4,5,1,3,2)$.

In this way, we form all permutations of $\{1,2,\ldots,n,n+1\}$ in a unique way. More formally, if we let $\mathcal{R}_n$ be the set of all permutations of $\{1,2,\ldots,n\}$, then this rule provides a bijection from $\mathcal{R}_n \times \{1,2,\ldots,n,n+1\}$ to $\mathcal{R}_{n+1}$. Therefore, we have $f(n+1) = (n+1)\cdot f(n)$ for all $n\in\mathbb{N}^+$. Combining this with the fact that $f(1)=1$, we conclude that $f(n)=n!$ for all $n\in\mathbb{N}^+$.

Definition 5.1.3. *Let A be a finite set with $|A|=n$, and let $k\in\mathbb{N}$ with $1\leq k\leq n$. A* partial permutation *of A of length k is an element of A^k with no repeated element. A partial permutation of length k is also called a* k-permutation *of A.*

Proposition 5.1.4. *If A is a finite set with $n\in\mathbb{N}^+$ elements and $k\in\mathbb{N}^+$ is such that $1\leq k\leq n$, then there are*

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

many k-permutations of A.

Proof. The proof is the same as for permutations, except we stop after k stages. Notice that the last term in the product, corresponding to the number of choices at stage k, is $n-(k-1) = n-k+1$ because at stage k we have chosen the first $k-1$ many element. Finally, notice that

$$\begin{aligned}\frac{n!}{(n-k)!} &= \frac{n(n-1)(n-2)\cdot(n-k+1)(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots 1}\\ &= n(n-1)(n-2)\cdots(n-k+1)\end{aligned}$$

giving the last equality. □

For example, using the standard 26-letter alphabet, there are $26\cdot 25\cdot 24 = \frac{26!}{23!} = 15,600$ many three-letter strings of letters having no repetition.

Notation 5.1.5. *If $k,n\in\mathbb{N}^+$ with $1\leq k\leq n$, we use the notation $(n)_k$ or $P(n,k)$ for the number of k-permutations of a set with n elements, i.e. we define*

$$(n)_k = P(n,k) = \frac{n!}{(n-k)!}.$$

We now count the number of functions between two finite sets, as well as the number of injections (it turns out that the number of surjections is much harder).

Proposition 5.1.6. *Suppose that A and B are finite sets with $|A| = m$ and $|B| = n$.*

1. *The number of functions from A to B is n^m.*
2. *If $m \leq n$, then the number of injective functions from A to B is $P(n, m) = \frac{n!}{(n-m)!}$.*

Proof. 1. To see this, first list the elements of A in some order as $a_1, a_2, \ldots, a_m$. A function assigns a unique value in B to each a_i, so we go through the a_i in order. For a_1, we have n possible images because we can choose any element of B. Once we've chosen this, we now have n possible images for a_2. As we go along, we always have n possible images for each of the a_i. Therefore, the number of functions from A to B is $n \cdot n \cdots n = n^m$.

2. Notice that if $n < m$, then there are no injective functions $f \colon A \to B$ by the Pigeonhole Principle. Suppose instead that $m \leq n$. The argument here is similar to the one for general functions, but we get fewer choices as we progress through A. As above, list the elements of A in some order as $a_1, a_2, \ldots, a_m$. A function assigns a unique value in B to each a_i, so we go through that a_i in order. For a_1, we have n possible images because we can choose any element of B. Once we've chosen this, we now have $n-1$ possible images for a_2 because we can choose any value of B other than the one we sent a_1 to. Then we have $n-2$ many choices for a_3, etc. Once we arrive at a_m, we have already used up $m-1$ many elements of B, so we have $n-(m-1) = n-m+1$ many choices for where to send a_m. Therefore, the number of functions from A to B is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-m+1) = \frac{n!}{(n-m)!},$$

which is $P(n, m)$.

□

Suppose we ask the following question: Given $A = \{1, 2, 3, 4, 5, 6, 7\}$, how many elements of A^4 contain the number 7 at least once? In other words, how many four digit numbers are there such that each digit is between 1 and 7 (inclusive), and 7 occurs at least once? A natural guess is that the answer is $4 \cdot 7^3$ by the following argument:

- First, pick one of the 4 positions to place the 7.
- Now we have three positions open. Going through them in order, we have 7 choices for what to put in each of these three positions.

This all looks great, but unfortunately there is a problem. It is indeed true that such a sequence of four choices does create one of the numbers we are looking for. If we choose the sequence $(3, 1, 5, 1)$, saying that we put a 7 in the third position and then place $(1, 5, 1)$ in order in the remaining positions, then we obtain the number 1571. However, the sequence of choices $(2, 7, 3, 4)$ and the sequence of choices $(1, 7, 3, 4)$ both produce the same string, namely 7734. More formally, the function that takes a position (for the first 7) together with a sequence of 3 digits, and produces the corresponding 4-digit sequence that contains a 7, is surjective but not injective. As a result, we do not have a bijection, and so can not count the set in this way.

In order to get around this problem, the key idea is to count the complement. That is, instead of counting the number of elements of A^4 that *do* contain the number 7 at least once, we count the number of elements of A^4 that *do not* contain the number 7 at all, and subtract this amount from the total number of elements in A^4. Now since $|A| = 7$, we have that $|A^4| = 7^4$ because we have 7 choices for each of the 4 spots. To count the number of elements of A^4 that do not contain a 7, we simply notice that we have 6 choices for

each of the 4 spots, so there are 6^4 of these. Therefore, by the Complement Rule, the number of elements of A^4 that do contain the number 7 at least once is $7^4 - 6^4$.

We next move on to a fundamental question that will guide a lot of our later work. Let $n \in \mathbb{N}^+$. We know that there are 2^n many subsets of $\{1, 2, \ldots, n\}$. However, what if we ask how many subsets there are of a certain size? For instance, how many subsets are there of $\{1, 2, 3, 4, 5\}$ that have exactly 3 elements? The intuitive idea is to make 3 choices: First, pick one of the 5 elements to go into our set. Next, pick one of the 4 remaining elements to add to it. Finally, finish off the process by picking one of the 3 remaining elements. For example, if we choose the number $1, 3, 5$ then we get the set $\{1, 3, 5\}$. Thus, a natural guess is that there are $5 \cdot 4 \cdot 3$ many subsets with 3 elements. However, recall that a set has neither repetition nor order, so just as in the previous example we count the same set multiple times. For example, picking the sequence $3, 5, 1$ would also give the set $\{1, 3, 5\}$. In fact, we arrive at the set $\{1, 3, 5\}$ in the following six ways:

$$\begin{array}{lll} (1,3,5) & (3,1,5) & (5,1,3) \\ (1,5,3) & (3,5,1) & (5,3,1) \end{array}$$

In hindsight, we realize that were just counting the number of 3-permutations of $\{1, 2, 3, 4, 5\}$, since the order matters there.

At this point, we may be tempted to throw our hands in the air as we did above. However, there is one crucial difference. In our previous example, some sequences of 4 numbers including a 7 were counted once (like 1571), some were counted twice (like 7712), and others were counted three or four times. However, in our current situation, *every* subset is counted exactly 6 times because given a set with 3 elements, we know that there are $3! = 6$ many permutations of that set (i.e. ways to arrange the elements of the set in order). The fact that we count each element 6 times means that the total number of subsets of $\{1, 2, 3, 4, 5\}$ having exactly 3 elements equals $\frac{5 \cdot 4 \cdot 3}{6} = 10$. The general principle that we are applying is the following:

Proposition 5.1.7 (Quotient Rule)**.** *Suppose that A is a finite set with $|A| = n$. Suppose that $\sim$ is an equivalence relation on A, and that every equivalence class has exactly k elements. In this case there are $\frac{n}{k}$ many equivalence classes.*

Proof. Let ℓ be the number of equivalence classes. To obtain an element of A, we can first pick one of the ℓ equivalence classes, and then pick one of the k many elements from that class. Since the equivalence classes partition A, it follows that this sequence of choices produces each element of A in a unique way. Thus, $n = k \cdot \ell$ by the Product Rule, and hence $\ell = \frac{n}{k}$. □

Another way to state the Quotient Rule is in terms of surjective functions.

Proposition 5.1.8 (Quotient Rule - Alternative Form)**.** *Suppose that A and B are a finite sets with $|A| = n$. Suppose that $f \colon A \to B$ is a surjective function and that $k \in \mathbb{N}^+$ has the property that*

$$|\{a \in A : f(a) = b\}| = k$$

for all $b \in B$ (i.e. every $b \in B$ is hit by exactly k elements of A). We then have $|B| = \frac{n}{k}$.

Proof. The argument is similar to previous proof. Let $\ell = |B|$. To obtain an element of A, we can first pick one of the ℓ elements of B, and then pick one of the k many elements from the set $\{a \in A : f(a) = b\}$. Notice that this sequence of choices produces each element of A in a unique way. Thus, $n = k \cdot \ell$ by the Product Rule, and hence $\ell = \frac{n}{k}$. □

In fact, these two versions of the Quotient Rule are really different aspects of the same underlying phenomena. To see this, given a surjective function $f \colon A \to B$, and define relation $\sim$ on A defined by letting $a_1 \sim a_2$ if $f(a_1) = f(a_2)$. It is then straightforward to check that $\sim$ is an equivalence relation on A, and that for all $c \in A$, we have

$$\overline{c} = \{a \in A : f(a) = f(c)\}.$$

In other words, the sets $\{a \in A : f(a) = b\}$ given in the second version are equivalence classes of $\sim$ (since f is surjective). If we also assume that each of these sets have the same size (as we do in the second version), then we are just saying that the equivalence classes have the same size, and hence we can apply the first version of the Quotient Rule.

Proposition 5.1.9. *Let $n, k \in \mathbb{N}^+$ and with $1 \leq k \leq n$. Suppose that A is a finite set with $|A| = n$. The number of subsets of A having exactly k elements equals*

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k! \cdot (n-k)!}.$$

Proof. We generalize the above argument. We know that the number of k-permutations of A equals

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Now a k-permutation of A picks k distinct elements of A, put also assigns an order to the elements. Now every subset of A of size k is coded by exactly $k!$ many such k-permutations because we can order the subset in $k!$ many ways. Therefore, by the Quotient Rule, the number of subsets of A having exactly k elements equals

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k! \cdot (n-k)!}.$$

□

Notice also that if $k = 0$, then there is one subset of any set having zero elements (namely $\emptyset$). Thus, by defining $0! = 1$, the above formula works in the case when $k = 0$ as well.

Definition 5.1.10. *Let $n, k \in \mathbb{N}$ and with $0 \leq k \leq n$. We define the notations $\binom{n}{k}$ and $C(n, k)$ by*

$$\binom{n}{k} = C(n, k) = \frac{n!}{k! \cdot (n-k)!}.$$

We call this the number of k-combinations *of an n-element set, and pronounce $\binom{n}{k}$ as "n choose k".*

For example, the number of 5-card poker hands from a standard 52-card deck is:

$$\binom{52}{5} = \frac{52!}{5! \cdot 47!} = 2,598,960.$$

We now give a number of examples of counting problems:

- Over the standard 26-letter alphabet, how many "words" of length 8 have exactly 5 consonants and 3 vowels? We build every such word in a unique way via a sequence of choices:
  - First, we pick out a subset of 3 of the 8 positions to house the vowels, and we have $\binom{8}{3}$ many possibilities.
  - Next, we pick 3 vowels in order allowing repetition to fill in these positions. Since we have 5 vowels, there are 5^3 many possibilities.
  - Finally, we pick 5 consonants in order, allowing repetition, to fill in remaining 5 positions. Since we have 21 consonants, there are 21^5 many possibilities.

  Since every word is uniquely determined by this sequence of choices, the number of such words is

$$\binom{8}{3} \cdot 5^3 \cdot 21^5 = 56 \cdot 5^3 \cdot 21^5.$$

- How many ways are there to seat n people around a circular table (so the only thing that matters is the relative position of people with respect to each other)? To count this, we use the Quotient Rule. We first consider each of the chairs as distinct. List the people in some order, and notice that we have n choices for where to seat the first person, then $n-1$ for where to seat the second, then $n-2$ for the third, and so forth. Thus, if the seats are distinct, then we have $n!$ many ways to seat the people. However, two such seating arrangements are equivalent if we can get one from the other via a rotation of the seats. Since there are n possible rotations, each seating arrangement occurs n times in this count, so the total number of such seatings is $\frac{n!}{n} = (n-1)!$.

  More formally, we can think about this as following. Consider all permutations of an n-element set (the people): we know that there are $n!$ of these. Now given two permutations, which are just sequences of length n without repetition, we consider two of these sequences equivalent exactly when every pair of numbers is the same distance apart where we allow "wrap around" (since the seating is circular). We then have that two such sequences are equivalent precisely when one is a cyclic shift of the other. Thus, every equivalence class has exactly n elements, and hence there are $\frac{n!}{n} = (n-1)!$ many circular arrangements.

- Suppose that we are in a city where all streets are straight and either east-west or north-south. Suppose that we are at one corner, and want to travel to a corner that is m blocks east and n blocks north, but we want to do it efficiently. More formally, we want to count the number of ways to get from the point $(0,0)$ to the point (m,n) where at each stage we either increase the x-coordinate by 1 or we increase the y-coordinate by 1. At first sight, it appears that we at each intersection, we have 2 choices: Either go east or go north. However, this is not really the case, because if we can east m times, then we are forced to go north the rest of the way. The idea for how to count this is that such a path is uniquely determined by a sequence of $m+n$ many E's and N's (representing east and north) having exactly m many E's. To determine such a sequence, we need only choose the positions of the m many E's, and there are

  $$\binom{m+n}{m}$$

  many choices. Of course, we could instead choose the positions of n many N's to count it as

  $$\binom{m+n}{n}$$

  which is the same number.

- How many anagrams (i.e. rearrangements of the letters) are there of MISSISSIPPI? Here is one approach. Notice that MISSISSIPPI has one M, four I's, four S's, and two P's, for a total of eleven letters. First pick the position of the M and notice that we have 11 choices. Once that is done, pick the position of the four I's and notice that this amount to picking a 4 element subset of the remaining 10 positions. There are $\binom{10}{4}$ many such choices. Once that is done, pick the position of the four S's and notice that this amount to picking a 4 element subset of the remaining 6 positions. There are $\binom{6}{4}$ many such choices. Once this is done, the position of the two P's is fixed. This gives a total number of anagrams equal to

  $$11\binom{10}{4}\binom{6}{4} = 11 \cdot \frac{10!}{4! \cdot 6!} \cdot \frac{6!}{4! \cdot 2!} = \frac{11!}{4! \cdot 4! \cdot 2!} = 34,650.$$

  Another argument is as follows. Think of distinguishing common letters with different colors. We then have 11! many ways to rearrange the letters, but this number overcounts the numbers of anagrams. Each actual anagram comes about in $4! \cdot 4! \cdot 2!$ many ways because we can permute the currently distinct four I's amongst each other in 4! ways, we can permute the currently distinct four S's amongst each

other in 4! ways, and we can permute the the currently distinct two P's amongst each other in 2! many ways. Thus, since each actual anagram is counted $4! \cdot 4! \cdot 2!$ many times in the 11! count, it follows that there are

$$\frac{11!}{4! \cdot 4! \cdot 2!} = 34,650$$

many anagrams of MISSISSIPPI.

As mentioned above, there are a total of

$$\binom{52}{5} = 2,598,960$$

many (unordered) 5-card poker hands from a standard 52-card deck. Using this, we now count the number of special hands of each type, as well as the probability of being dealt such a hand (this probability is calculated by dividing the number of such hands by the the total number $2,598,960$). We use the fact that each card has one of four suits (clubs, diamonds, hearts, and spades) and one of thirteen ranks (2, 3, 4, 5, 6, 7, 8, 9,10, jack, queen, king, ace). We follow the common practice of allowing the ace to be either a low card or a high card for a straight, but we do not allow "wrap around" straights such as king, ace, 2, 3, 4.

- Straight Flush: There are

  $$4 \cdot 10 = 40$$

  many of these because they are determined by picking the suit and then picking the rank of the lowest card (from ace through 10). The probability is about .00154%.

- Four of a kind: There are

  $$13 \cdot 48 = 624$$

  of these because we choose a rank (and take all four cards of that rank), and then choose one of the remaining 48 cards. The probability is about .0256%.

- Full House: There are

  $$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3,744$$

  many, which can be seen by making the following sequence of choices:

  - Choose one of the 13 ranks for the three of a kind.
  - Choose 3 of the 4 suits for the three of a kind.
  - Choose one of the 12 remaining ranks for the pair.
  - Choose 2 of the 4 suits for the pair.

  The probability is about .14406%.

- Flush: There are

  $$4 \cdot \binom{13}{5} = 5,148$$

  many because we need to choose 1 of the 4 suits, and then 5 of the 13 ranks. However, 40 of these are actually straight flushes, so we really have $5,108$ many flushes that are not stronger hands. The probability is about .19654%

- Straight: There are
$$10 \cdot 4^5 = 10,240$$
many because we need to choose the rank of the lowest card, and the suits for the five cards in increasing order of rank. However, we again have that 40 of these are straight flushes, so we really have $10,200$ many straights that are not stronger hands. The probability is about .39246%.

- Three of a kind: There are
$$13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2 = 54,912$$
many, which can be seen by making the following sequence of choices:
  - Choose one of the 13 ranks for the three of a kind.
  - Choose 3 of the 4 suits for the three of a kind.
  - Choose two of the other ranks for the remaining two cards (they are different because we do not want to include full houses).
  - Choose the suit of the lower ranked card not in the three of a kind.
  - Choose the suit of the higher ranked card not in the three of a kind.

  (Alternatively, we can choose the last two cards in different ranks in $48 \cdot 44$ many ways, but then we need to divide by 2 because the order of choosing these does not matter.) The probability is about 2.1128%.

- Two Pair: There are
$$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 44 = 123,552$$
many, which can be seen by making the following sequence of choices:
  - Choose the two ranks for the two pairs.
  - Choose the two suits for the lower ranked pair.
  - Choose the two suits for the higher ranked pair.
  - Choose one of the 44 cards not in these two ranks.

  The probability is about 4.7539%.

- One pair: There are
$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240$$
many, which can be seen by making the following sequence of choices:
  - Choose the rank for the pair.
  - Choose the two suits for the pair.
  - Choose three distinct ranks for the other three cards (which are not the same rank as the pair).
  - Choose the suit of the lowest ranked card not in the pair.
  - Choose the suit of the middle ranked card not in the pair.
  - Choose the suit of the highest ranked card not in the pair.

  The probability is about 42.257%.

5.2 The Binomial Theorem and Properties of Binomial Coefficients

Recall that if $n, k \in \mathbb{N}$ with $k \leq n$, then we defined

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

Notice that when $k = n = 0$, then $\binom{n}{k} = 1$ because we define $0! = 1$, and indeed there is a unique subset of $\emptyset$ having 0 elements, namely $\emptyset$. When $n, k \in \mathbb{N}$ with $n < k$, then we define

$$\binom{n}{k} = 0$$

because there are no subsets of an n-element set with cardinality k (notice that the above formula doesn't make sense because $n - k < 0$).

Using Proposition 4.2.4, we know that whenever $k, n \in \mathbb{N}$ are such that $k \leq n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$

because the function that takes the relative complement is a bijection between subsets of cardinality k and subsets of cardinality $n - k$. Of course, one can prove this directly from the formulas because

$$\begin{aligned}\binom{n}{n-k} &= \frac{n!}{(n-k)! \cdot (n-(n-k))!} \\ &= \frac{n!}{(n-k)! \cdot k!} \\ &= \frac{n!}{k! \cdot (n-k)!} \\ &= \binom{n}{k}.\end{aligned}$$

Although the algebraic manipulations here are easy, the bijective proof feels more satisfying because it "explains" the formula. Proving that two numbers are equal by showing that the both count the numbers of elements in one common set, or by proving that there is a bijection between a set counted by the first number and a set counted by the second, is called either a *combinatorial proof* or a *bijective proof*.

Proposition 5.2.1. *Let $n, k \in \mathbb{N}^+$ with $0 < k < n$. We have*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. One extremely unenlightening proof is to expand out the formula on the right and do terrible algebraic manipulations on it. If you haven't done so, I encourage you to do it. However, we use the combinatorial description of $\binom{n}{k}$ to give a more meaningful combinatorial argument. Let $n, k \in \mathbb{N}$ with $k \leq n$. Consider a set A with n many elements. To determine $\binom{n}{k}$, we need to count the number of subsets of A of size k. We do this as follows. Fix an arbitrary $a \in A$. Now an arbitrary subset of A of size k fits into exactly one of the following types:

- The subset has a as an element. In this case, to completely determine the subset, we need to pick the remaining $k - 1$ elements of the subset from $A \backslash \{a\}$. Since $A \backslash \{a\}$ has $n - 1$ elements, the number of ways to do this is $\binom{n-1}{k-1}$.

- The subset does not have a as an element. In this case, to completely determine the subset, we need to pick all k elements of the subset from $A\backslash\{a\}$. Since $A\backslash\{a\}$ has $n-1$ elements, the number of ways to do this is $\binom{n-1}{k}$.

Since every subset of A of size k fits into exactly one these types, we have written the collection of all such subsets as a disjoint union (of those satisfying the first condition, and those satisfying the second). By the Sum Rule, we conclude that the number of subsets of A of size k equals $\binom{n-1}{k-1} + \binom{n-1}{k}$. □

Using this proposition, together with the fact that

$$\binom{n}{0} = 1 \qquad \text{and} \qquad \binom{n}{n} = 1$$

for all $n \in \mathbb{N}$, we can compute $\binom{n}{k}$ recursively to obtain the following table. The rows are labeled by n and the columns by k. To determine the number that belongs in a given square, we simply add the number above it and the number above and to the left. This table is known as *Pascal's Triangle*:

$\binom{n}{k}$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0
3	1	3	3	1	0	0	0	0
4	1	4	6	4	1	0	0	0
5	1	5	10	10	5	1	0	0
6	1	6	15	20	15	6	1	0
7	1	7	21	35	35	21	7	1

There are many curious properties of Pascal's Triangle that we will discover in time. One of the first things to note is that these numbers seem to appear in other places. For example, if $x, y \in \mathbb{R}$, then we have:

- $(x+y)^1 = x + y$
- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Looking at these, it appears that the coefficients are exactly the corresponding elements of Pascal's Triangle. What is the connection here? Notice that if we do not use commutativity and do not collect like terms (so just use distributivity repeatedly), we have

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= x(x+y) + y(x+y)\\ &= xx + xy + yx + yy,\end{aligned}$$

and so

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(xx + xy + yx + yy)\\ &= x(xx + xy + yx + yy) + y(xx + xy + yx + yy)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy.\end{aligned}$$

In other words, it looks like when we fully expand $(x+y)^n$, without using commutativity or collecting x's and y's, then we are getting a sum of all sequences of x's and y's of length n. Thus, if we want to know the coefficient of $x^{n-k}y^k$, then we need only ask how many such sequences have exactly k many y's (or equivalently exactly $n-k$ many x's), and the answer is $\binom{n}{k} = \binom{n}{n-k}$ because we need only pick out the position of the y's (or the x's). More formally, we can prove this by induction.

Theorem 5.2.2 (Binomial Theorem)**.** *Let $x, y \in \mathbb{R}$ and let $n \in \mathbb{N}^+$. We have*

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \\ &= \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\end{aligned}$$

Proof. We prove the result by induction. When $n = 1$, we trivially have

$$(x+y)^1 = x + y = \binom{1}{0}x + \binom{1}{1}y$$

Suppose then that we have an $n \in \mathbb{N}^+$ for which we know that the statement is true. We then have

$$\begin{aligned}(x+y)^{n+1} &= (x+y)^n \cdot (x+y) \\ &= \left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n\right)\cdot(x+y) \\ &= \left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n\right)\cdot x \\ &\qquad \left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n\right)\cdot y \\ &= \binom{n}{0}x^{n+1} + \binom{n}{1}x^n y + \binom{n}{2}x^{n-1}y^2 + \cdots + \binom{n}{n-1}x^2y^{n-1} + \binom{n}{n}xy^n \\ &\qquad + \binom{n}{0}x^n y + \binom{n}{1}x^{n-1}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-1} + \binom{n}{n-1}xy^n + \binom{n}{n}y^{n+1} \\ &= x^{n+1} + \left(\binom{n}{1} + \binom{n}{0}\right)\cdot x^n y + \left(\binom{n}{2} + \binom{n}{1}\right)\cdot x^{n-1}y^2 + \cdots + \left(\binom{n}{n} + \binom{n}{n-1}\right)\cdot xy^n + y^{n+1} \\ &= \binom{n+1}{0}x^{n+1} + \binom{n+1}{1}x^n y + \binom{n+1}{2}x^{n-1}y^2 + \cdots + \binom{n+1}{n}xy^n + \binom{n+1}{n+1}y^{n+1},\end{aligned}$$

where we have used Proposition 5.2.1 to combine each of the sums to get the last line. □

Corollary 5.2.3. *For any $n \in \mathbb{N}^+$, we have*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Proof 1. We use the Binomial Theorem in the special case where $x = 1$ and $y = 1$ to obtain

$$\begin{aligned} 2^n &= (1+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot 1^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}. \end{aligned}$$

This completes the proof. □

Proof 2. Let $n \in \mathbb{N}^+$ be arbitrary. We give a combinatorial proof by arguing that both sides count the number of subsets of an n-element set. Suppose then that A is a set with $|A| = n$. On the one hand, we know that $|\mathcal{P}(A)| = 2^n$ by Corollary 4.2.3.

We now argue that

$$|\mathcal{P}(A)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

For each $k \in \mathbb{N}$ with $0 \leq k \leq n$, let $\mathcal{P}_k(A)$ be the subset of $\mathcal{P}(A)$ consisting of those subsets of A having exactly k elements. We then have that

$$\mathcal{P}(A) = \mathcal{P}_0(A) \cup \mathcal{P}_1(A) \cup \mathcal{P}_2(A) \cup \cdots \cup \mathcal{P}_n(A)$$

and furthermore that the $\mathcal{P}_k(A)$ are pairwise disjoint (i.e. if $k \neq \ell$, then $\mathcal{P}_k(A) \cap \mathcal{P}_\ell(A) = \emptyset$). Therefore,

$$|\mathcal{P}(A)| = |\mathcal{P}_0(A)| + |\mathcal{P}_1(A)| + |\mathcal{P}_2(A)| + \cdots + |\mathcal{P}_n(A)|$$

by the General Sum Rule. Now for each k with $0 \leq k \leq n$, we know that

$$|\mathcal{P}_k(A)| = \binom{n}{k},$$

so it follows that

$$|\mathcal{P}(A)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Hence

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

because both sides count the number of elements of $\mathcal{P}(A)$. □

Corollary 5.2.4. *For any $n \in \mathbb{N}^+$, we have*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

Thus

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Proof 1. We use the Binomial Theorem in the special case where $x = 1$ and $y = -1$ to obtain

$$\begin{aligned} 0 &= 0^n \\ &= (1 + (-1))^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot (-1)^k \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}. \end{aligned}$$

This gives the first claim. Adding $\binom{n}{k}$ to both sides for each odd k, we conclude that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots.$$

Since

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

by the previous result, it follows that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots.$$

□

Proof 2. Let $n \in \mathbb{N}^+$ be arbitrary. We begin by giving a combinatorial proof for the second claim. We first show that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}.$$

Let A be an arbitrary set with $|A| = n$, and list the elements of A as $A = \{a_1, a_2, \ldots, a_n\}$. Recall that we know that $|\mathcal{P}(A)| = 2^n$ because for each i, we have 2 choices for whether or not to include a_i in our subset. Now in our case, the sum on the left

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots$$

counts the numbers of subset of A having an even number of elements. We argue that 2^{n-1} also counts the number of subsets of A having an even number of elements. To build these subsets, we make the following sequence of choices:

- Determine whether to include a_1 in our subset: We have 2 choices.
- Determine whether to include a_2 in our subset: We have 2 choices.
- ...
- Determine whether to include a_{n-1} in our subset: We have 2 choices.
- Finally, examine the first $n - 1$ choices, and determine whether we have included an even number of a_i. If so, do not include a_n in our subset. If not, include a_n in our subset.

Notice that in the last step, we do not make any choices, but do one of two things that are completely determined by the previous choices. Now no matter what sequence of choices we make, we end up with a subset of A having an even number of elements, and furthermore every subset with an even number of elements arrises in a unique way. Since there are 2 choices in each of the opening $n-1$ stages, it follows that there are 2^{n-1} many subsets of A with an even number of elements. Therefore,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$$

Now the proof that

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$$

is completely analogous except for changing the last stage (or alternatively comes from the complement rule). Finally, since both of these sums equals 2^{n-1}, we conclude that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

□

Proposition 5.2.5. *For any $n, k \in \mathbb{N}^+$ with $k \leq n$, we have*

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$$

hence

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}.$$

Proof. We claim that each side counts the number of ways of selecting a committee consisting of k people, including a distinguished president of the committee, from a group of n people. On the one hand, we can do this as follows:

- First pick the committee of k people from the total group of all n people. We have $\binom{n}{k}$ many ways to do this.
- Within this committee, choose one of the k people to serve as president. We have k options here.

Therefore, the number of possibilities is $k \cdot \binom{n}{k}$. On the other hand, we can count it as follows.

- First pick one of the n people to be the president.
- Next pick the remaining $k-1$ many people to serve on the committee amongst the remaining $n-1$ people. We have $\binom{n-1}{k-1}$ many ways to do this.

Therefore, the number of possibilities is $n \cdot \binom{n-1}{k-1}$.

Since each side counts the number of elements of one set, the values must be equal. Therefore,

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}.$$

□

Proposition 5.2.6. *For any n, we have*

$$\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}.$$

Proof 1. We have

$$\begin{aligned}
\sum_{k=1}^{n} k \cdot \binom{n}{k} &= \sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} && \text{(by Proposition 5.2.5)} \\
&= n \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} \\
&= n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} \\
&= n \cdot 2^{n-1} && \text{(by Corollary 5.2.3)}
\end{aligned}$$

□

Proof 2. We give a direct combinatorial proof by arguing that both sides count the number of ways of building a committee, including a distinguished president of that committee, of any size from a group of n people.

On the one hand we can count the number of such committees as follows. We break up the situation into cases based on the size of the committee. For a committee of size k including a distinguished president, we know from Proposition 5.2.5 that there are $k \cdot \binom{n}{k}$ many ways to do this. Since we can break up the collection of all such committees into the pairwise disjoint union of those committees of size 1, those of size 2, etc. Therefore, by the Sum Rule, the number of ways to do this is $\sum_{k=1}^{n} k \cdot \binom{n}{k}$.

On the other hand, we can count the number of such committees differently. First, pick the president of the committee, and notice that we have n choices. Once we pick the president, we need to pick the rest of the committee. Thus, we need to pick a subset (of any size) from the remaining $n-1$ people to fill out the committee, and we know that there are 2^{n-1} many subsets of a set of size $n-1$. Therefore, there are $n \cdot 2^{n-1}$ many such committees.

Since each side counts the number of elements of one set, the values must be equal. Therefore,

$$\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}.$$

□

Proof 3. We give another proof using the Binomial Theorem, which tells us that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

for all $x, y \in \mathbb{R}$. Plugging in $y = 1$, we conclude that

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

for all $x \in \mathbb{R}$. Now each side is a function of the real variable x, so taking the derivative of each side, it follows that

$$n(x+1)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}$$

for all $x \in \mathbb{R}$. Plugging in $x = 1$, we conclude that

$$n \cdot 2^{n-1} = \sum_{k=1}^{n} k \cdot \binom{n}{k}$$

This completes the proof. □

Proposition 5.2.7. *If $k \leq n$, then*

$$\sum_{m=k}^{n}\binom{m}{k} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

and since $\binom{m}{k} = 0$ if $m < k$, it follows that

$$\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1}.$$

Proof 1. Using Proposition 5.2.1 repeatedly, we have:

$$\begin{aligned}
\binom{n+1}{k+1} &= \binom{n}{k} + \binom{n}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-1}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \binom{n-2}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k+2}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k+1}{k} + \binom{k+1}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k+1}{k} + \binom{k}{k}.
\end{aligned}$$

where the last line follows from the fact that

$$\binom{k+1}{k+1} = 1 = \binom{k}{k}.$$

□

Proof 2. We can also give a combinatorial proof by arguing that

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k}$$

counts the number of subsets of $[n+1] = \{1, 2, 3, \ldots, n, n+1\}$ having cardinality $k+1$. To see this, first notice that if $A \subseteq [n+1]$ with $|A| = k+1$, then the largest element of A must be at least $k+1$. For each $m \in \mathbb{N}$ with $k+1 \leq m \leq n+1$, let

$$\mathcal{F}_m = \{A \in \mathcal{P}([n+1]) : |A| = k+1 \text{ and } \max(A) = m\}.$$

We then have that the $\mathcal{F}_m$ are pairwise disjoint, and that

$$\mathcal{F}_{k+1} \cup \mathcal{F}_{k+2} \cup \cdots \cup \mathcal{F}_n \cup \mathcal{F}_{n+1}$$

equals the collection of subsets of $[n+1]$ having cardinality $k+1$. Using the General Sum Rule, it follows that

$$|\mathcal{F}_{k+1}| + |\mathcal{F}_{k+2}| + \cdots + |\mathcal{F}_n| + |\mathcal{F}_{n+1}| = \binom{n+1}{k+1}.$$

Now notice that for any $m \in \mathbb{N}$ with $k+1 \leq m \leq n+1$, we have

$$|\mathcal{F}_m| = \binom{m-1}{k}$$

because to determine any $A \in \mathcal{F}_m$, we need only choose the k elements of A that are less than the maximum value m. Therefore,

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}.$$

□

Plugging in $k = 1$, we get

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}.$$

for all $n \in \mathbb{N}^+$. Since $\binom{m}{1} = m$ for all $m \in \mathbb{N}^+$, it follows that

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

for all $n \in \mathbb{N}^+$. Notice that letting $k = 2$, we conclude that that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$$

for all $n \in \mathbb{N}^+$. Since $\binom{1}{2} = 0$, we can also write this as

$$\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}.$$

Now we can use these to find a formula for the sum of the first n squares:

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

The idea is to find $A, B \in \mathbb{R}$ such that

$$m^2 = A \cdot \binom{m}{1} + B \cdot \binom{m}{2}$$

is true for all $m \in \mathbb{N}^+$, because if we can do this, then we can use the above summation formulas for the two sums that appear on the right. Since $\binom{m}{1} = m$ for all $m \in \mathbb{N}^+$, and

$$\binom{m}{2} = \frac{m(m-1)}{2}$$

for all $m \in \mathbb{N}^+$ (even for $m = 1$ because then both sides are 0), we want to find A and B such that:

$$m^2 = A \cdot m + B \cdot \frac{m(m-1)}{2}$$

for all $m \in \mathbb{N}^+$. Now

$$\begin{aligned} A \cdot m + B \cdot \frac{m(m-1)}{2} &= A \cdot m + B \cdot \frac{m^2 - m}{2} \\ &= \left(A - \frac{B}{2}\right) \cdot m + \frac{B}{2} \cdot m^2 \end{aligned}$$

so equating coefficients with $m^2 = 0 \cdot m + 1 \cdot m^2$, we want to solve the linear system:

$$\begin{array}{rcrcl} A & - & \frac{1}{2} \cdot B & = & 0 \\ & & \frac{1}{2} \cdot B & = & 1 \end{array}$$

Now $A = 1$ and $B = 2$ as the unique solution to this system, so it follows that

$$m^2 = \binom{m}{1} + 2 \cdot \binom{m}{2}$$

is true for all $m \in \mathbb{N}^+$. Thus, using Proposition 5.2.7, we conclude that

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 &= \left[\binom{1}{1} + 2 \cdot \binom{1}{2}\right] + \left[\binom{2}{1} + 2 \cdot \binom{2}{2}\right] + \cdots + \left[\binom{n}{1} + 2 \cdot \binom{n}{2}\right] \\ &= \left[\binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1}\right] + 2 \cdot \left[\binom{1}{2} + \binom{2}{2} + \cdots + \binom{n}{2}\right] \\ &= \binom{n+1}{2} + 2 \cdot \binom{n+1}{3} \\ &= \frac{(n+1)n}{2} + 2 \cdot \frac{(n+1)n(n-1)}{6} \\ &= \frac{3(n+1)n}{6} + \frac{2(n+1)n(n-1)}{6} \\ &= \frac{n(n+1)(3+2n-2)}{6} \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

One can generalize these techniques to get the sum of the first n cubes. Doing so would require finding $A, B, C \in \mathbb{R}$ such that

$$m^3 = A \cdot \binom{m}{1} + B \cdot \binom{m}{2} + C \cdot \binom{m}{3}$$

for all $m \in \mathbb{N}^+$. Although it's not too onerous to do the algebra in order to set up the linear system, and then solve for A, B, C, we will see more unified ways to determine these coefficients (along with for fourth powers, etc.) soon.

Suppose that we want to pick out 5 days from the month of February (having 28 days) in such a way that we do not pick two consecutive days. How can we count it? Although we want to pick out an unordered subset, one idea is to first count the number of *ordered* choices, and then divide by 5!. The idea then is to pick out one day, and we have 28 choices. Once we've picked that day out, we then pick out a second day. It may appear that we have 25 choices here because we've eliminated one day and it's two neighbors. However, that it is only true if we did not pick out the first or last days of February in our first choice. Thus, the number of options in round two depends on our choice from round one. You might think about counting those sets including the first and/or last days of February as special cases, but this doesn't solve all of the problems. For example, if we choose 11 and 18 in our first two rounds, then we've eliminated 6 days and have 22 choices for the third round. However, if we choose 11 and 13 in our first two rounds, then we've only eliminated 5 days and so have 23 choices for the third round. In other words, we need a new way to count this.

Let's attack the problem from a different angle. Instead of trying to avoid bad configurations directly, we think about picking out an arbitrary subset of 5 days and "spreading" them to guarantee that the result will not have any consecutive days. To do this, we will leave the lowest numbered day alone, but add 1 to the second lowest day (to ensure we have a "gap" between the first two), and then add 2 to the middle day, etc. More formally, given an arbitrary subset $\{a_1, a_2, a_3, a_4, a_5\}$ of [28] with $a_1 < a_2 < a_3 < a_4 < a_5$, we turn it into the subset $\{a_1, a_2+1, a_3+2, a_4+3, a_5+4\}$ which does not have any consecutive days. The only problem is that now we might "overflow". For example, although

$$\{3, 4, 15, 16, 21\} \mapsto \{4, 6, 17, 19, 25\}$$

works out just fine, we also have

$$\{1, 10, 21, 26, 27\} \mapsto \{1, 11, 23, 30, 31\}$$

which is not allowed. However, there's an easy fix. Instead of picking our original subset from [28], we pick it from [24], for a total of $\binom{24}{5}$ many possibilities. In general, we have the following:

Proposition 5.2.8. *The number of subsets of* $[n] = \{1, 2, 3, \ldots, n\}$ *of size* k *having no two consecutive numbers equals* $\binom{n-k+1}{k}$.

Proof. We establish a bijection between the k-element subsets of $[n-k+1]$ and the sets we want. Given a subset $\{a_1, a_2, a_3, \ldots, a_k\}$ of $[n-k+1]$ with $a_1 < a_2 < a_3 < \cdots < a_k$, we map it to the set $\{a_1, a_2+1, a_3+2, \ldots, a_k+(k-1)\}$, i.e. the i^{th} element of the new set equals $a_i + (i-1)$. Now since $a_i < a_{i+1}$ for each i, we have that $a_{i+1} - a_i \geq 1$ for each i, and hence

$$\begin{aligned} a_{i+1} + ((i+1)-1) - (a_i + (i-1)) &= a_{i+1} + i - a_i - i + 1 \\ &= a_{i+1} - a_i + 1 \\ &\geq 1 + 1 \\ &= 2 \end{aligned}$$

for all i, so there are no consecutive elements in the resulting set. Furthermore, since $a_k \leq n-k+1$, we have $a_k + (k-1) \leq n-k+1+(k-1) = n$, so the resulting subset is indeed a subset of $[n]$ of size k having no two consecutive elements. Notice that this function is injective because if $\{a_1, a_2+1, a_3+2, \ldots, a_k+(k-1)\} = \{b_1, b_2+1, b_3+2, \ldots, b_k+(k-1)\}$, then $a_i+(i-1) = b_i+(i-1)$ for all i, hence $a_i = b_i$ for all i. Furthermore, given a subset $\{c_1, c_2, c_3, \ldots, c_k\}$ of $[n]$ with $c_1 < c_2 < c_3 < \cdots < c_n$ and $c_{i+1} - c_i \geq 2$ for all i, we have that $\{c_1, c_2-1, c_3-2, \ldots, c_k-(k-1)\}$ is a subset of $[n-k+1]$ that maps to $\{c_1, c_2, c_3, \ldots, c_k\}$, so it is surjective. The result follows. □

What if we just wanted to count the number number of subsets of $[n]$ having no two consecutive numbers, without any size restrictions? One approach is to sum over all possible sizes to obtain:

$$\sum_{k=0}^{n} \binom{n-k+1}{k} = \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots + \binom{1}{n}$$

Of course, many of the terms on the right equal 0 because if $k > n-k+1$, i.e. if $k > \frac{n+1}{2}$, then $\binom{n-k+1}{k} = 0$. Thus, if we let $\lfloor m \rfloor$ be the greatest integer less than or equal to m, then we have

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k}.$$

For example, the number of subsets of $[6] = \{1,2,3,4,5,6\}$ having no two consecutive numbers is

$$\begin{aligned}\sum_{k=0}^{3} \binom{7-k}{k} &= \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} \\ &= 1 + 6 + 10 + 4 \\ &= 21\end{aligned}$$

while the number of subsets of $[7] = \{1,2,3,4,5,6,7\}$ having no two consecutive numbers is

$$\begin{aligned}\sum_{k=0}^{4} \binom{8-k}{k} &= \binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4} \\ &= 1 + 7 + 15 + 10 + 1 \\ &= 34.\end{aligned}$$

Notice that we are summing up diagonals of Pascal's triangle, and we are seeing Fibonacci numbers. You will prove that this holds true generally on the homework.

Returning to the Binomial Theorem, what happens if we look powers of $x+y+z$ instead of $x+y$? For example, we have

$$\begin{aligned}(x+y+z)^2 &= (x+y+z)(x+y+z) \\ &= x(x+y+z) + y(x+y+z) + z(x+y+z) \\ &= xx + xy + xz + yx + yy + yz + zx + zy + zz\end{aligned}$$

Thus, we obtain a sum of $9 = 3 \cdot 3$ terms, where each term is an ordered product of two elements (with repetition) from $\{x,y,z\}$. If we work out $(x+y+z)^3$, we see a sum of $27 = 3 \cdot 3 \cdot 3$ terms, where each possible ordered sequence of 3 elements (with repetition) from $\{x,y,z\}$ appears exactly once. In general, one expects that we expand $(x+y+z)^n$, then we obtain a sequence of 3^n many terms where each possible ordered sequence of n elements (with repetition) from $\{x,y,z\}$ appears exactly once. What happens when we use collapse these sums by using commutativity, so write $xxz + xzx + zxx$ as $3x^2z$? In general, we are asking what the coefficient of $x^a y^b z^c$ will be in the result? Notice that we need only examine the coefficients where $a+b+c = n$ because each term involves a product of n of the variables. Suppose then that $a+b+c = n$. To know the coefficient of $x^a y^b z^c$, we want to know the number of sequences of x's, y's, and z's of length n having exactly a many x's, b many y's, and c many z's. To count these, we can first pick out that a positions in which to place the x's in $\binom{n}{a}$ many ways. Next, we have $n-a$ open positions, and need to pick out b positions to place the y's in $\binom{n-a}{b}$ many ways. Finally, we have $n-a-b = c$ many positions for the c many

z's, so they are completely determined. Thus, if $a+b+c=n$, then the coefficient of $x^a y^b z^c$ in $(x+y+z)^n$ equals

$$\begin{aligned}\binom{n}{a}\cdot\binom{n-a}{b} &= \frac{n!}{a!\cdot(n-a)!}\cdot\frac{(n-a)!}{b!\cdot(n-a-b)!}\\ &= \frac{n!}{a!\cdot b!\cdot(n-a-b)!}\\ &= \frac{n!}{a!\cdot b!\cdot c!}.\end{aligned}$$

More generally, suppose that we expand $(x_1+x_2+\cdots+x_k)^n$. In the result, we will have a sum of term of the form $x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}$ where the $a_i\in\mathbb{N}$ and $a_1+a_2+\cdots+a_k=n$. To determine the coefficient of such a term, we need only determine the number of sequences of x_i of length n such that there are exactly a_1 many x_1's, exactly a_2 many x_2's, ..., and exactly a_k many x_k's. Following the above template, the number of such sequences equals

$$\begin{aligned}&\binom{n}{a_1}\cdot\binom{n-a_1}{a_2}\cdot\binom{n-a_1-a_2}{a_3}\cdots\binom{n-a_1-a_2-\cdots-a_{k-2}}{a_{k-1}}\cdot\binom{n-a_1-a_2-\cdots-a_{k-1}}{a_k}\\ &\quad=\binom{n}{a_1}\cdot\binom{n-a_1}{a_2}\cdot\binom{n-a_1-a_2}{a_3}\cdots\binom{n-a_1-a_2-\cdots-a_{k-2}}{a_{k-1}}\cdot\binom{a_k}{a_k}\\ &\quad=\frac{n!}{a_1!\cdot(n-a_1)!}\cdot\frac{(n-a_1)!}{a_2!\cdot(n-a_1-a_2)!}\cdot\frac{(n-a_1-a_2)!}{a_3!\cdot(n-a_1-a_2-a_3)!}\cdots\frac{(n-a_1-a_2-\cdots-a_{k-1})!}{a_{k-1}!\cdot(n-a_1-a_2-\cdots-a_{k-2}-a_{k-1})!}\\ &\quad=\frac{n!}{a_1!\cdot a_2!\cdot a_3!\cdots a_{k-1}!\cdot(n-a_1-a_2-\cdots-a_{k-2}-a_{k-1})!}\\ &\quad=\frac{n!}{a_1!\cdot a_2!\cdot a_3!\cdots a_{k-1}!\cdot a_k!}\end{aligned}$$

Notice that this is just like our problem with anagrams of MISSISSIPPI. Instead of doing the above count, we could have treated all the x_1 as different (and x_2 as different, etc.), rearranged them in $n!$ many ways, and then divided by the overcount from the permuting the x_i within themselves in $a_1!$ ways, the x_2 within themselves in $a_2!$ many ways, etc.

Definition 5.2.9. *If $n, a_1, a_2, \ldots, a_k\in\mathbb{N}$ and $a_1+a_2+\cdots+a_k=n$, we define*

$$\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1!\cdot a_2!\cdots a_k!}$$

We call this a multinomial coefficient.

The above argument proves the generalization of the Binomial Theorem:

Theorem 5.2.10 (Multinomial Theorem)**.** *For all $n, k\in\mathbb{N}^+$, we have*

$$(x_1+x_2+\cdots+x_k)^n = \sum\binom{n}{a_1, a_2, \ldots, a_k}x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}$$

where the sum is taken over all k-tuples of nonnegative integers $(a_1, a_2, \ldots, a_k)$ such that $a_1+a_2+\cdots+a_k=n$.

5.3 Compositions and Partitions

Compositions

There are six different M&M colors: Red, Yellow, Blue, Green, Orange, Brown. Suppose that we want to pick out 13 total M&M's. How ways can you do it, if all that matters is how many of each color we take?

Notice that we can model this as follows: if we let a_i be the number that you choose with color i, then we need $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 13$.

Definition 5.3.1. *Let $n, k \in \mathbb{N}$. A sequence of nonnegative integers $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$ is called a* weak composition *of n into k parts. If all the a_i are positive, then it is called a* composition *of n into k parts.*

For example $(1, 3, 5, 3)$ is a composition of 12 into 4 parts and $(2, 0, 5, 1, 0, 0)$ is a weak composition of 8 into 6 parts.

One can view the number of weak compositions of n into k parts as the number of ways to distribute n identical balls into k distinct boxes. In this interpretation, the value a_i is the number of balls that we put into box i. We are treating the balls as identical because all that matters are the number of balls in each box, but the boxes are distinct because $(5, 2, 1)$ is different than $(2, 5, 1)$.

We can also view these another way. Recall that a k-permutation of n distinct objects is a way to pick out k of those objects, where order matters and repetition is not allowed. Also, a k-combination of n distinct objects is a way to pick out k of those objects, where order does not matter and repetition is not allowed. A different way to interpret a weak composition of n into k parts is as a way to pick out n objects from k distinct objects, where order doesn't matter but repetition *is* allowed (yes, the n and k have switched, and this is incredibly annoying). The value a_i is the number of times that we pick out object i. Due to the fact that order doesn't matter but repetition is allowed, some sources think about something they call *multisets*. The idea is to allow one to write something like "$\{1, 1, 4\}$" and think about it as different from "$\{1, 4\}$", but the same as "$\{1, 4, 1\}$". Since, by definition, two sets are equal exactly when they have the same elements, we should introduce new notation rather than { and } used in sets. Instead of dealing with all of these issues, we avoid the situation entirely by writing $(2, 0, 0, 1)$ to represent that we picked the number 1 twice and the number 4 once.

The number of weak compositions of n into k parts is the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n,$$

while the number of compositions of n into k parts is the number of positive integer solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

How do we count the number of weak compositions of n into k parts? In the M&M case, think about lining them up in order of color, so red first, then yellow, etc. If we eliminate the colors from the M&M's themselves, then we only need some kind of "marker" to distinguish when we change colors. If we represent the M&M's as dots, then we can place 5 bars to denote the dividing line as to when we switch colors. Since we have 5 bars and 13 M&M's that we have to put into a line, we have 18 positions and need to choose the positions for the 5 bars. Therefore, there are $\binom{18}{5}$ many possibilities.

Proposition 5.3.2. *Let $n, k \in \mathbb{N}$. The number of weak compositions of n into k parts is*

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Proof. As above, there is a bijection between arrangements of n dots and $k-1$ bars into a line and weak compositions of n into k parts (the number of dots before the first bar is a_1, then number of dots between the first and second is a_2, etc.). We want to place $n + k - 1$ many objects, and we need only choose the $k - 1$ positions for the bars (or alternatively the n positions for the dots). Therefore, the number of weak compositions of n into k parts equals

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

□

Another way to visualize this is as follows: Consider the following bijection between subsets of $[n+k-1]$ of size n and weak compositions of n into k parts: Given a subset $\{a_1, a_2, \ldots, a_n\}$ of $[n+k-1]$ with $a_1 < a_2 < \cdots < a_n$, consider the multiset "$\{a_1, a_2-1, a_3-2, \ldots, a_n-(n-1)\}$" and form the corresponding weak composition. For example if $k=5$ and $n=3$, then $n+k-1=7$ and we do the following:

$$\{1,2,3\} \mapsto \text{“}\{1,1,1\}\text{”} \mapsto (3,0,0,0,0)$$
$$\{1,3,7\} \mapsto \text{“}\{1,2,5\}\text{”} \mapsto (1,1,0,0,1)$$
$$\{3,4,6\} \mapsto \text{“}\{3,3,4\}\text{”} \mapsto (0,0,2,1,0)$$

More formally, given a subset $\{a_1, a_2, \ldots, a_n\}$ of $[n+k-1]$ with $a_1 < a_2 < \cdots < a_n$, we send it to the sequence $(b_1, b_2, \ldots, b_k)$ where b_ℓ equals the number of i such that $a_i - (i-1) = \ell$, i.e. the cardinality of the set $\{i : a_i = i + \ell - 1\}$.

Now that we've determined the number of *weak* compositions of n into k parts, we can count the number of compositions of n into k parts. The idea is that if $k \leq n$, then the number of positive integer solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n$$

equals to the number of nonnegative solutions to

$$x_1 + x_2 + \cdots + x_k = n - k.$$

Corollary 5.3.3. *Let $n, k \in \mathbb{N}$ with $k \leq n$. The number of compositions of n into k parts equals*

$$\binom{n-1}{k-1} = \binom{n-1}{n-k}.$$

Proof. First distribute one ball to each of the k boxes . We now have $n-k$ balls to put into k boxes with no restrictions, and so we want to count the number of weak compositions of $n-k$ into k parts. The answer to this is:

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$$

Since $(n-1)-(k-1) = n-k$, this also equals

$$\binom{n-1}{n-k}.$$

More formally, given a weak composition $(a_1, a_2, \ldots, a_k)$ of $n-k$ into k parts, the sequence $(a_1+1, a_2+1, \ldots, a_k+1)$ is composition of n into k parts, and this mapping is a bijection. □

Another way to visualize the previous corollary with a direct bijection is as follows: Consider the function

$$(a_1, a_2, \ldots, a_k) \mapsto \{a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_{k-1}\}$$

from compositions of n into k parts to $(k-1)$-element subsets of $[n-1]$. For example if $n=10$ and $k=4$, then

$$(1,2,3,4) \mapsto \{1,3,6\}$$
$$(6,1,1,2) \mapsto \{6,7,8\}$$
$$(2,1,1,6) \mapsto \{2,3,4\}$$

Notice that since $a_i \geq 1$ for all i, we have $a_1 < a_1+a_2 < \cdots < a_1+a_2+\cdots+a_{k-1}$. Now since $a_1+a_2+\cdots+a_k = n$ and $a_k \geq 1$, it follows that $a_1+a_2+\cdots+a_{k-1} \leq n-1$, and hence the set on the right

is an element of $[n-1]$. Finally, one must check that this is a bijection, but I'll leave that to you (since we already have a proof of the result).

What happens if we try to count *all* compositions of a number n without specifying the number of parts? For example, we have 4 compositions of 3 given by (3), $(1,2)$, $(2,1)$, and $(1,1,1)$. The compositions of 4 are (4), $(1,3)$, $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, and $(1,1,1,1)$, so we have 8 of those.

Theorem 5.3.4. *The number of compositions of n is 2^{n-1}.*

Proof. We give two proofs. The first is to notice that a composition of n must be a composition of n into k parts for some unique k with $1 \leq k \leq n$. Therefore, the number of compositions of n equals

$$\begin{aligned}\sum_{k=1}^{n} \binom{n-1}{k-1} &= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \\ &= 2^{n-1}. \qquad \text{(by Corollary 5.2.3)}\end{aligned}$$

Alternatively, we can give a direct combinatorial proof. Write down n dots. Notice that we can not put a bar before the first dot or after the last one, and we also can not put two bar in the same place because in a composition all numbers must be positive. Therefore, a composition arises by picking a subset of the $n-1$ spaces between the dots to serve as bars (i.e. the dividers). Since there are 2^{n-1} many subsets of a set with $n-1$ many elements, it follows that there are 2^{n-1} many compositions of n. □

Set Partitions

Above we considered the case where the balls were identical and the boxes were distinct. Now consider the case where the balls are distinct but the boxes are identical.

Definition 5.3.5. *A* (set) partition *of a set A is a set $\{B_1, B_2, \ldots, B_k\}$ where the B_i are nonempty pairwise disjoint subsets of A with*

$$A = B_1 \cup B_2 \cup \cdots \cup B_k.$$

In this case, we call this a partition of A into k nonempty parts.

Definition 5.3.6. *Given $n, k \in \mathbb{N}^+$ with $k \leq n$, we define $S(n,k)$ to be the number of partitions of $[n]$ into k nonempty parts. The numbers $S(n,k)$ are called the* Stirling numbers of the second kind *and are sometimes denoted by:*

$$S(n,k) = \left\{ {n \atop k} \right\}.$$

We also define $S(0,0) = 1$, $S(n,0) = 0$ if $n \geq 1$, and $S(n,k) = 0$ if $k > n$.

For example, we have $S(3,2) = 3$ because the following are all possible partitions of $[3] = \{1,2,3\}$ into 2 parts:

- $\{\{1\},\{2,3\}\}$
- $\{\{2\},\{1,3\}\}$
- $\{\{3\},\{1,2\}\}$

Notice that these are all of them because if we partition [3] into 2 parts, then one must have size 1 and the other have size 2, so the partition is completely determined by the choice of the the element that is in its own block (and hence there are $\binom{3}{1} = 3$ many choices).

Here are few more examples:

- If $n \geq 1$, then
$$S(n,1) = 1 = S(n,n)$$
because the only partition on $[n]$ into one part is $\{\{1,2,3,\ldots,n\}\}$ and the only partition into n parts is $\{\{1\},\{2\},\ldots,\{n\}\}$.

- We have $S(4,3) = \binom{4}{2} = 6$ because such a partition must have one set of size 2 and the others of size 1, so we need only choose the subset of size 2.

- More generally, for any $n \geq 2$, we have
$$S(n,n-1) = \binom{n}{2}$$
because a partition of $[n]$ into $n-1$ many blocks must have one block of size 2 and $n-2$ of size 1, so we need to pick the two unique elements for the block of size 2.

- The number $S(4,2)$ is more interesting. We can partition $\{1,2,3,4\}$ into a set of size 3 and a set of size 1, or into two sets of size 2. There are $\binom{4}{1} = 4$ ways to do the former because we need only pick the element in the set of size 1. For the latter, there are 3 possibilities:
  - $\{\{1,2\},\{3,4\}\}$
  - $\{\{1,3\},\{2,4\}\}$
  - $\{\{1,4\},\{2,3\}\}$

  Therefore, $S(4,2) = 4 + 3 = 7$.

In general, the numbers $S(n,k)$ are difficult to compute. Recall that
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
whenever $k, n \in \mathbb{N}^+$. We get a similar recurrence here.

Theorem 5.3.7. *For all $k, n \in \mathbb{N}^+$ with $k \leq n$, we have*
$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k).$$
In other words, if $k \leq n$, then
$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \cdot \left\{ {n-1 \atop k} \right\}.$$

Proof. A partition of $[n]$ into k parts is of one of two possible types:

- *Case 1:* The number n is in a block by itself. If we remove the block $\{n\}$, then we are left with a partition of $[n-1]$ into $k-1$ parts, so there are $S(n-1,k-1)$ many such possibilities. Notice that every partition of $[n]$ into k blocks having $\{n\}$ as one of the blocks arises in a unique way from such a partition of $[n-1]$ into $k-1$ parts. Thus, there are $S(n-1,k-1)$ many partitions of this type.

- *Case 2:* The number n is not in its own block. If we remove n from its block, we then obtain a partition of $[n-1]$ into k parts, and there are $S(n-1,k)$ many possible outcomes. Notice that each of these outcomes arise in k many ways because given a partition of $[n-1]$ into k blocks, we can add n into any of the blocks to obtain a partition of $[n]$ into k parts. Therefore, there are $k \cdot S(n-1,k)$ many partitions of this type.

It follows that $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$. □

We now get a triangle like Pascal's triangle, but with $S(n,k) = \left\{ {n \atop k} \right\}$ in place of $C(n,k) = \binom{n}{k}$.

$\left\{ {n \atop k} \right\}$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	1	3	1	0	0	0	0
4	0	1	7	6	1	0	0	0
5	0	1	15	25	10	1	0	0
6	0	1	31	90	65	15	1	0
7	0	1	63	301	350	140	21	1

Given $n, k \in \mathbb{N}^+$, recall that we have the following from Proposition 5.1.6:

- The number of functions $f\colon [n] \to [k]$ equals k^n because for each $i \in [n]$, we have k possibilities for the value of $f(i)$.
- If $k < n$, then there are no injective functions $f\colon [n] \to [k]$ by the Pigeonhole Principle.
- If $k > n$, then the number of injective functions $f\colon [n] \to [k]$ equal $k(k-1)(k-2)\cdots(k-n+1) = (k)_n = \frac{k!}{(k-n)!}$ because we have k choices for the value of $f(1)$, then $k-1$ for the value of $f(2)$, ..., and finally $k-(n-1)$ for the value of $f(n)$.

We now give one way to count the number of surjective functions from $[n]$ to $[k]$ in terms of Stirling numbers.

Proposition 5.3.8. *Given $n, k \in \mathbb{N}^+$, there are exactly $k! \cdot S(n,k)$ many surjective functions $f\colon [n] \to [k]$.*

Proof. If $k > n$, then there are no surjective functions $f\colon [n] \to [k]$, and $k! \cdot S(n,k) = k! \cdot 0 = 0$. Suppose then that $k \leq n$. Consider a surjective $f\colon [n] \to [k]$. For each $c \in [k]$, let $B_c = \{a \in [n] : f(a) = c\}$, i.e. B_c is the set of all elements of $[n]$ than map to c. Since f is surjective, we know that $B_c \neq \emptyset$ for all $c \in [k]$. Furthermore, since f is a function, the sets $B_1, B_2, \ldots, B_k$ are pairwise disjoint, and $[n] = B_1 \cup B_2 \cup \cdots \cup B_k$. Therefore, $\{B_1, B_2, \ldots, B_k\}$ is a partition of $[n]$ into k nonempty parts. Notice that each of these partitions arise in $k!$ many ways because we can reorder the B_i in terms of their outputs, i.e. if $n = 4$ and $k = 2$ then $\{\{1,4\},\{2,3\}\}$ is a partition arising from both the function

$$f(1) = 1 \quad f(2) = 2 \quad f(3) = 2 \quad f(4) = 1$$

and the function

$$f(1) = 2 \quad f(2) = 1 \quad f(3) = 1 \quad f(4) = 2.$$

In other words, every surjective function arises uniquely from a partition of $[n]$ into k nonempty parts, together with a permutation of $[k]$. Therefore, the number of surjective functions $f\colon [n] \to [k]$ equals $k! \cdot S(n,k)$. □

Theorem 5.3.9. *For all $m, n \in \mathbb{N}^+$, we have*

$$m^n = \sum_{k=1}^{n} k! \cdot S(n,k) \cdot \binom{m}{k},$$

i.e.

$$m^n = \sum_{k=1}^{n} k! \cdot \left\{ {n \atop k} \right\} \cdot \binom{m}{k}.$$

Proof. The left-hand side m^n is simply the number of functions from $[n]$ to $[m]$ (by Proposition 5.1.6). The key fact is that given any function $f\colon A \to B$, if we let $C = \text{range}(f)$, then we can view f as a surjective function $f\colon A \to C$. Thus, every function $f\colon [n] \to [m]$ can be viewed as a surjective function onto some nonempty subset of $[m]$. Now given a subset $X \subseteq [m]$ with $|X| = k$, we know from the previous proposition that there are $k! \cdot S(n,k)$ many surjections from $[n]$ to X. For a fixed k, there are $\binom{m}{k}$ many subsets of $[m]$ of size k, so there are $\binom{m}{k} \cdot k! \cdot S(n,k)$ many functions from $[n]$ to $[m]$ whose range has size k. Summing over all possible sizes for the range, we conclude that the number of functions from $[n]$ to $[m]$ equals

$$\sum_{k=0}^{n} \binom{m}{k} \cdot k! \cdot S(n,k)$$

Therefore,

$$m^n = \sum_{k=0}^{n} k! \cdot S(n,k) \cdot \binom{m}{k}.$$

□

In particular, we have

$$\begin{aligned} m^2 &= 1 \cdot 1 \cdot \binom{m}{1} + 2 \cdot 1 \cdot \binom{m}{2} \\ &= 1 \cdot \binom{m}{1} + 2 \cdot \binom{m}{2} \end{aligned}$$

for all $m \in \mathbb{N}$ as we learned above. We also have

$$\begin{aligned} m^3 &= 1 \cdot 1 \cdot \binom{m}{1} + 2 \cdot 3 \cdot \binom{m}{2} + 6 \cdot 1 \cdot \binom{m}{3} \\ &= 1 \cdot \binom{m}{1} + 6 \cdot \binom{m}{2} + 6 \cdot \binom{m}{3} \end{aligned}$$

and

$$\begin{aligned} m^4 &= 1 \cdot 1 \cdot \binom{m}{1} + 2 \cdot 7 \cdot \binom{m}{2} + 6 \cdot 6 \cdot \binom{m}{3} + 24 \cdot 1 \cdot \binom{m}{4} \\ &= 1 \cdot \binom{m}{1} + 14 \cdot \binom{m}{2} + 36 \cdot \binom{m}{3} + 24 \cdot \binom{m}{4} \end{aligned}$$

for all $m \in \mathbb{N}$. Using these formulas together with Proposition 5.2.7, we can now develop formulas for the

sum of the first n cubes, the first n fourth powers, etc. For example, we have

$$\begin{aligned}
\sum_{m=1}^{n} m^3 &= \sum_{m=1}^{n} \left(1 \cdot \binom{m}{1} + 6 \cdot \binom{m}{2} + 6 \cdot \binom{m}{3}\right) \\
&= \sum_{m=1}^{n} \binom{m}{1} + 6 \cdot \sum_{m=1}^{n} \binom{m}{2} + 6 \cdot \sum_{m=1}^{n} \binom{m}{3} \\
&= \binom{n+1}{2} + 6 \cdot \binom{n+1}{3} + 6 \cdot \binom{n+1}{4} \\
&= \frac{(n+1)n}{2} + 6 \cdot \frac{(n+1)n(n-1)}{6} + 6 \cdot \frac{(n+1)n(n-1)(n-2)}{24} \\
&= \frac{(n+1)n}{2} + (n+1)n(n-1) + \frac{(n+1)n(n-1)(n-2)}{4} \\
&= \frac{(n+1)n}{4} \cdot (2 + 4(n-1) + (n-1)(n-2)) \\
&= \frac{(n+1)n}{4} \cdot (2 + 4n - 4 + n^2 - 3n + 2) \\
&= \frac{(n+1)n}{4} \cdot (n^2 + n) \\
&= \frac{(n+1)^2 n^2}{4} \\
&= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

Therefore, we obtain the surprising result that

$$\sum_{m=1}^{n} m^3 = \left(\sum_{m=1}^{n} m\right)^2$$

for all $n \in \mathbb{N}^+$.

Definition 5.3.10. *Let $n \in \mathbb{N}$. The number of all partitions of $[n]$ into nonempty parts is denoted by $B(n)$ and is called the n^{th} Bell number. We also define $B(0) = 0$. Notice that*

$$B(n) = \sum_{k=0}^{n} S(n,k) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

for all $n \in \mathbb{N}$.

Recall than an equivalence relation on A induces a partition of A into nonempty parts through the equivalence classes. Conversely, it's not hard to show that if $\{B_1, B_2, \ldots, B_k\}$ is a partition of A with each $B_i \neq \emptyset$, then the relation $a \sim b$ if there exists an i with $a, b \in B_i$ is an equivalence relation on A whose equivalence classes are the B_i. Therefore, $B(n)$ equals the number of equivalence relations on a set of size n.

Adding up the rows of the above table, we obtain the following values for the first few Bell numbers:

n	$B(n)$
0	1
1	1
2	2
3	5
4	15
5	52
6	203
7	877

Theorem 5.3.11. *For any $n \in \mathbb{N}$, we have*

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} \cdot B(k).$$

Proof. We need to argue that the right-hand side counts the number of partitions of $[n+1]$. We look at the block containing $n+1$. We examine how many elements are *not* in the block containing $n+1$. If there are k such elements, then there are $\binom{n}{k}$ many ways to choose these elements (and hence choose the $n-k$ many elements of $[n]$ grouped with $n+1$) and then $B(k)$ many ways to partition them. Thus,

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} \cdot B(k).$$

Alternatively, we can count as follows. If the block containing $n+1$ has exactly k elements, then there are $\binom{n}{k-1}$ many ways to choose the other elements in the block, and then $B(n+1-k)$ many ways to partition the rest. Thus

$$\begin{aligned} B(n+1) &= \sum_{k=1}^{n+1} \binom{n}{k-1} \cdot B(n+1-k) \\ &= \sum_{k=0}^{n} \binom{n}{k} \cdot B(n-k) \\ &= \sum_{k=0}^{n} \binom{n}{n-k} \cdot B(k) \\ &= \sum_{k=0}^{n} \binom{n}{k} \cdot B(k). \end{aligned}$$

□

Integer Partitions

We've seen that compositions correspond to ways to distribute n identical balls to k distinct boxes in such a way that each box receives at least one ball. Also, (set) partitions correspond to ways to distribute n distinct balls to k identical boxes in such a way that each box receives at least one ball. We now introduce (integer) partitions that correspond to ways to distribute n identical balls to k identical boxes in such a way that each box receives at least one ball.

Definition 5.3.12. *An* (integer) partition *of an $n \in \mathbb{N}$ into k parts is a composition $(a_1, a_2, \ldots, a_k)$ of n where $a_1 \geq a_2 \geq \cdots \geq a_k$. The number of partitions of n into k parts is denoted by $p(n,k)$. We also define $p(0,0) = 1$.*

Notice that $p(n,0) = 0$ if $n \geq 1$ and $p(n,k) = 0$ if $k > n$. We have $p(4,2) = 2$ because $(2,2)$ and $(3,1)$ are the only partitions of 4 into 2 parts. Notice that $p(7,3) = 1$ because the partitions are $(5,1,1)$, $(4,2,1)$, $(3,3,1)$, and $(3,2,2)$.

Definition 5.3.13. *The number of partitions of n (into any number of parts) is denoted by $p(n)$, so*

$$p(n) = \sum_{k=0}^{n} p(n,k).$$

In order to calculate p, we first establish a simple recurrence like we did for for $\binom{n}{k}$ and $S(n,k)$.

Theorem 5.3.14. *For all $n, k \in \mathbb{N}$ with $0 < k < n$, we have*

$$\begin{aligned} p(n,k) &= \sum_{i=1}^{k} p(n-k,i) \\ &= p(n-k,1) + p(n-k,2) + \cdots + p(n-k,k). \end{aligned}$$

Proof. Given a partition of n into k parts, if we subtract 1 from each part, then we obtain a partition of $n-k$ into some number (at most k) parts. Notice that we might have fewer parts, because any 1 in the original partition will become 0. However, since $k < n$, and we are subtracting k, at least one part will remain. Moreover, it's straightforward to check that this is a bijection (i.e. that it is injective and that every partition of $n-k$ into at most k parts arise). □

Thus, we have

$$p(7,3) = p(4,1) + p(4,2) + p(4,3)$$

and

$$\begin{aligned} p(7,4) &= p(3,1) + p(3,2) + p(3,3) + p(3,4) \\ &= p(3,1) + p(3,2) + p(3,3). \end{aligned}$$

By starting with some simple values, we can use this recurrence to fill in a table of values.

$*$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	1	1	1	0	0	0	0
4	0	1	2	1	1	0	0	0
5	0	1	2	2	1	1	0	0
6	0	1	3	3	2	1	1	0
7	0	1	3	4	3	2	1	1

Adding up the rows, we obtain the following values:

n	$p(n)$
0	1
1	1
2	2
3	3
4	5
5	7
6	11
7	15

The question of how fast $p(n)$ grows is extremely interesting and subtle. It turns out that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

where $\exp(x) = e^x$ and $f(n) \sim g(n)$ means that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

5.4 Inclusion-Exclusion

Recall that if A and B are any finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

What about three sets, i.e. if we wanted to count $|A \cup B \cup C|$? A natural guess would be that we need to subtract off the various intersection, so one might hope that $|A \cup B \cup C|$ equals

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|.$$

Let's examine if this is correct. Notice that if $x \in A$, but $x \notin B$ and $x \notin C$, then x contributes 1 to $|A \cup B \cup C|$, and in the formula it contributes

$$1 + 0 + 0 - 0 - 0 - 0 = 1.$$

Similar arguments work if x is in only B, or x is in only C. Let's examine what happens if x is in two of the sets, say $x \in A$, $x \in B$, but $x \notin C$. Again, x contributes 1 to $|A \cup B \cup C|$, and in the formula it contributes

$$1 + 1 + 0 - 1 - 0 - 0 = 1.$$

Again, everything looks good so far. Finally, suppose that x is an element of each of A, B, and C. As usual, x contributes 1 to $|A \cup B \cup C|$, but in the formula it contributes

$$1 + 1 + 1 - 1 - 1 - 1 = 0.$$

Thus, elements that are if $A \cap B \cap C$ are not counted at all on the right-hand side. To correct this, we need to add it back in. We then claim that the correct formula is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Working though each of the possibilities, one can check that this count is correct to matter where x lies in the Venn diagram of sets. How does this generalize? For four sets, one can show by working through all of the cases that

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ & - |A_1 \cap A_2 \cap A_3 \cap A_4|. \end{aligned}$$

It's extremely tedious to check all the possibilities here, so we would like a way to prove that this works in general. We'll do that below, but first we'll demonstrate how to use these formulas to count an interesting set. For our example, we will count the number of primes less than or equal to 120. Before jumping into this, we prove a few small but important facts.

Proposition 5.4.1. *Let $n \in \mathbb{N}^+$ with $n \geq 2$. If n is not prime, then there is a prime p such that $p \mid n$ and $p \leq \sqrt{n}$.*

Proof. Suppose that $n \geq 2$ is not prime. Since n is not prime, we can fix $d \in \mathbb{N}$ with $1 < d < n$ such that $d \mid n$. Fix $c \in \mathbb{Z}$ with $cd = n$. Notice that $c > 0$ because both $d > 0$ and $n > 0$, and moreover we must have $1 < c < n$ (if $c = 1$ then $d = n$, and if $c = n$ then $d = 1$). Now at least one of $c \leq \sqrt{n}$ or $d \leq \sqrt{n}$ must be true, because otherwise $n = cd > \sqrt{n} \cdot \sqrt{n} = n$. In either case, this number has a prime divisor (by Proposition 3.2.2) less than or equal to it, so by transitivity of divisibility, n has a prime divisor p with $p \leq \sqrt{n}$. □

Proposition 5.4.2. *If $a, b, c \in \mathbb{Z}$ are such that $a \mid c$, $b \mid c$, and $\gcd(a, b) = 1$, then $ab \mid c$.*

Proof. Exercise. □

Proposition 5.4.3. *Let $a \in \mathbb{Z}$ and let $p_1, p_2, \ldots, p_k$ be distinct primes. If $p_i \mid a$ for all i, then $p_1 p_2 \cdots p_k \mid a$.*

Proof. We prove the result by induction on k. Notice that if $k = 1$, then the statement is trivial. Suppose that we know the statement is true for a fixed $k \in \mathbb{N}$. Let $p_1, p_2, \ldots, p_k, p_{k+1}$ be distinct primes with the property that $p_i \mid a$ for all i. By induction, we know that $p_1 p_2 \cdots p_k \mid a$. We also have that $p_{k+1} \mid a$ by assumption. Using Corollary 3.2.11, we know that $\gcd(p_1 p_2 \cdots p_k, p_{k+1}) = 1$, so Proposition 5.4.2 allows us to conclude that $p_1 p_2 \cdots p_k p_{k+1} \mid a$. This completes the induction. □

We now return to counting the number of primes in $[120]$. By Proposition 5.4.1, if $a \in [120]$ is not prime and $a \geq 2$, then a is divisible by some prime less than or equal to $\sqrt{a} \leq \sqrt{120}$. Now $\sqrt{120} < 11$, so the only primes less than or equal to $\sqrt{120}$ are 2, 3, 5, and 7. We thus let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and $p_4 = 7$. For each i, let A_i be the set of numbers in $[120]$ divisible by p_i. We count

$$|A_1 \cup A_2 \cup A_3 \cup A_4|,$$

which is the number of elements of $[120]$ that are divisible by at least one 2, 3, 5, or 7. We have

$$|A_1| = \frac{120}{2} = 60 \qquad |A_2| = \frac{120}{3} = 40 \qquad |A_3| = \frac{120}{5} = 24 \qquad |A_4| = \left\lfloor \frac{120}{7} \right\rfloor = 17.$$

To determine the cardinalities of intersections, we use Proposition 5.4.3. For example, the numbers divisible by both 2 and 3 are the numbers divisible by 6. Working these out, we conclude that

$$|A_1 \cap A_2| = \frac{120}{6} = 20 \qquad |A_1 \cap A_3| = \frac{120}{10} = 12 \qquad |A_1 \cap A_4| = \left\lfloor \frac{120}{14} \right\rfloor = 8$$

$$|A_2 \cap A_3| = \frac{120}{15} = 8 \qquad |A_2 \cap A_4| = \left\lfloor \frac{120}{21} \right\rfloor = 5 \qquad |A_3 \cap A_4| = \left\lfloor \frac{120}{35} \right\rfloor = 3.$$

Next we compute

$$|A_1 \cap A_2 \cap A_3| = \frac{120}{30} = 4 \qquad |A_1 \cap A_2 \cap A_4| = \left\lfloor \frac{120}{42} \right\rfloor = 2$$

$$|A_1 \cap A_3 \cap A_4| = \frac{120}{70} = 1 \qquad |A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{120}{105} \right\rfloor = 1$$

and

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{120}{210} \right\rfloor = 0.$$

Thus

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = (60 + 40 + 24 + 17) - (20 + 12 + 8 + 8 + 5 + 3) + (4 + 2 + 1 + 1) - 0 = 93.$$

By the Complement Rule, it follows that there are

$$120 - 93 = 27$$

many numbers in [120] that are not divisible by any of 2, 3, 5, or 7. All of these except 1 are prime, so this gives 26 new primes in [120]. Adding back in the primes 2, 3, 5, and 7, we see that there are 30 primes in [120].

Theorem 5.4.4 (Inclusion-Exclusion)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets. We then have*

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{S \in \mathcal{P}([n]) \setminus \{\emptyset\}} (-1)^{|S|-1} \cdot |\bigcap_{i \in S} A_i| \\ &= \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq [n], |S|=k} |\bigcap_{i \in S} A_i|. \end{aligned}$$

Less formally, this says that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots.$$

Proof. Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$ be arbitrary. Let $T = \{i \in [n] : x \in A_i\}$, i.e. T is the nonempty set of indices i such that $x \in A_i$. Let $k = |T|$ and notice that $k \geq 1$. We examine the number of times that x is counted on each side. On the left, x contributes 1 to the cardinality. On the right, it contributes

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \cdots + (-1)^{k-1}\binom{k}{k}$$

to the sum. Now from Corollary 5.2.4, we know that

$$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots - (-1)^{k}\binom{k}{k} = 0.$$

Hence

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \cdots + (-1)^{k-1}\binom{k}{k} = \binom{k}{0} = 1$$

Therefore, every $x \in A_1 \cup A_2 \cup \cdots \cup A_n$ contributes 1 to both sides. The result follows. □

We next count the number of surjections $f\colon [n] \to [k]$. Of course we know that the answer is $k! \cdot S(n,k)$ from Proposition 5.3.8, but we count it in a different way using Inclusion-Exclusion (from which we will be able to derive a formula for $S(n,k)$). We first illustrate the general argument in the special case where $n = 7$ and $k = 4$, i.e. we count the number of surjections $f\colon [7] \to [4]$. The idea is to count the complement. We know that there are 4^7 many total functions $f\colon [7] \to [4]$, so we count the number of functions that are *not* surjective. Now a function can fail to be a surjective by missing 1, missing 2, missing 3, or missing 4. Thus, given $i \in [4]$, we let A_i be the set of functions $f\colon [7] \to [4]$ such that $i \notin \text{range}(f)$. Then the set of functions $f\colon [7] \to [4]$ that are not surjective equals $A_1 \cup A_2 \cup A_3 \cup A_4$. Now we know that:

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ & - |A_1 \cap A_2 \cap A_3 \cap A_4|. \end{aligned}$$

To count $|A_1|$, we need to count the number of functions $f\colon [7] \to [4]$ such that $1 \notin \text{range}(f)$. This is just the number of functions $f\colon [7] \to \{2,3,4\}$, which equals 3^7. Similarly, $|A_2| = |A_3| = |A_4| = 3^7$. To count $|A_1 \cap A_2|$, we just need to count the number of functions $f\colon [7] \to [4]$ such that $1, 2 \notin \text{range}(f)$. This is just the number of functions $f\colon [7] \to \{3,4\}$, which equals 2^7. Following through on this, we conclude that

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & 3^7 + 3^7 + 3^7 + 3^7 \\ & - 2^7 - 2^7 - 2^7 - 2^7 - 2^7 - 2^7 \\ & + 1^7 + 1^7 + 1^7 + 1^7 + 1^7 \\ & - 0,\end{aligned}$$

so

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4 \cdot 3^7 - 6 \cdot 2^7 + 4 \cdot 1^7.$$

Notice that coefficients are $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, and $\binom{4}{3} = 4$ because $\binom{4}{m}$ is the number of ways to pick out m elements from $[4]$. It follows that the number of surjective functions $f\colon [7] \to [4]$ equals

$$4^7 - (4 \cdot 3^7 - 6 \cdot 2^7 + 4 \cdot 1^7) = 4^7 - 4 \cdot 3^7 + 6 \cdot 2^7 - 4 \cdot 1^7 = 8,400.$$

We now generalize this argument.

Theorem 5.4.5. *Let $n, k \in \mathbb{N}^+$ with $k \leq n$. The number of surjections $f\colon [n] \to [k]$ is*

$$\sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n.$$

Proof. The total number of functions $f\colon [n] \to [k]$ is k^n. For each $i \in [k]$, let A_i be the set of all functions $f\colon [n] \to [k]$ such that $i \notin \text{range}(f)$. We then have that

$$A_1 \cup A_2 \cup \cdots \cup A_k$$

is the set of all functions which are *not* surjective, and we count

$$|A_1 \cup A_2 \cup \cdots \cup A_k|$$

using Inclusion-Exclusion. Let $S \subseteq [k]$ be arbitrary, and let $m = |S|$. We then have that

$$\bigcap_{i \in S} A_i$$

is the set of functions whose range is contained in $[k] \backslash S$, so since $|[k] \backslash S| = k - m$, it follows that

$$|\bigcap_{i \in S} A_i| = (k - |S|)^n = (k-m)^n.$$

Therefore

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_k| &= \sum_{S \subseteq [k] \backslash \{\emptyset\}} (-1)^{|S|-1} \cdot |\bigcap_{i \in S} A_i| \\ &= \sum_{m=1}^{k} (-1)^{m-1} \sum_{S \subseteq [k], |S|=m} |\bigcap_{i \in S} A_i| \\ &= \sum_{m=1}^{k} (-1)^{m-1} \sum_{S \subseteq [k], |S|=m} (k - |S|)^n \\ &= \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} (k-m)^n,\end{aligned}$$

where the last line follows from the fact that $\binom{k}{m}$ is the number of subsets of $[k]$ of cardinality m. Thus, the number of surjections $f\colon [n] \to [k]$ is

$$k^n - \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} (k-m)^n = k^n + \sum_{m=1}^{k} (-1)^m \binom{k}{m} (k-m)^n$$
$$= \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n.$$

□

Corollary 5.4.6. *Let $n, k \in \mathbb{N}^+$ with $k \leq n$. We have*

$$S(n,k) = \frac{1}{k!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n$$
$$= \sum_{m=0}^{k} (-1)^m \frac{(k-m)^n}{m! \cdot (k-m)!}.$$

Proof. We know that the number of surjections $f\colon [n] \to [k]$ equals $k! \cdot S(n,k)$ by Proposition 5.3.8, and it also equals

$$\sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n$$

by Theorem 5.4.5. Therefore,

$$k! \cdot S(n,k) = \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n,$$

and hence

$$S(n,k) = \frac{1}{k!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n$$
$$= \sum_{m=0}^{k} (-1)^m \frac{(k-m)^n}{m! \cdot (k-m)!}.$$

□

For example, since

$$\sum_{m=0}^{4} (-1)^m \binom{4}{m} (4-m)^7 = 8,400$$

from above, we have

$$S(7,4) = \frac{8,400}{24} = 350.$$

Definition 5.4.7. *A* derangement *of $[n]$ is a permutation $(a_1, a_2, \ldots, a_n)$ of $[n]$ such that $a_i \neq i$ for all i.*

For example, $(3,1,4,2)$ is a derangement of $[4]$, but $(3,2,4,1)$ is not (because $a_2 = 2$).

Theorem 5.4.8. *Let $n \in \mathbb{N}^+$. The number of derangements of $[n]$ is*

$$n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Proof. We know that there are $n!$ many permutations of $[n]$. For each $i \in [n]$, let A_i be the set of all permutations $(a_1, a_2, \ldots, a_n)$ of $[n]$ such that $a_i = i$. We then have that

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

is the set of all functions which are *not* derangements. We count

$$|A_1 \cup A_2 \cup \cdots \cup A_n|$$

using Inclusion-Exclusion. Let $S \subseteq [n]$ be arbitrary, and let $k = |S|$. We then have that

$$\bigcap_{i \in S} A_i$$

is the set of permutations of $[n]$ such that $a_i = i$ for all $i \in S$. To count this, notice that k of the elements are determined, and the remaining $n - k$ elements can be permuted in the remaining $n - k$ spots arbitrarily, so

$$|\bigcap_{i \in S} A_i| = (n - |S|)! = (n - k)!.$$

We then have

$$\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{S \subseteq [n] \setminus \{\emptyset\}} (-1)^{|S|-1} \cdot |\bigcap_{i \in S} A_i| \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq [n], |S|=k} |\bigcap_{i \in S} A_i| \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq [n], |S|=k} (n-k)! \\
&= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! \\
&= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} \\
&= n! \cdot \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}.
\end{aligned}$$

Thus, the number of derangements of $[n]$ is

$$n! - n! \cdot \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

□

Notice that the fraction of permutations that are derangements equals

$$\begin{aligned}
\sum_{k=0}^{n} \frac{(-1)^k}{k!} &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \\
&= \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.
\end{aligned}$$

For example, when $n = 6$, we have

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{53}{144} \approx .36806,$$

so approximately 36.8% of the permutations are derangements. When $n = 7$, we have

$$\frac{53}{144} - \frac{1}{5040} = \frac{1854}{5040} = \frac{103}{280} \approx .36786,$$

so again about 36.8% of the permutations are derangements. Now if you've seen infinite series, then you know that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

for all $x \in \mathbb{R}$. In particular, when $x = -1$, we have

$$\begin{aligned} e^{-1} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \\ &= 1 - (-1) + \frac{(-1)^2}{2!} - \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} - \dots \\ &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots . \end{aligned}$$

Therefore, as n gets large, the percentage of permutations of $[n]$ that are derangements approaches the number

$$1/e \approx .36788.$$

Chapter 6

Graph Theory

6.1 Graphs, Multigraphs, Representations, and Subgraphs

Definition 6.1.1. *A* graph *G is a pair (V, E) of sets such that:*

- *V is a nonempty set.*
- *E is a (possibly empty) set such that each element is subset of V of cardinality 2. In other words, $E \subseteq \mathcal{P}_2(V)$.*

Elements of V are called vertices*, and elements of E are called* edges*. We say that G is finite if V is finite (in which case E must be finite as well).*

For example, if we let

$$V = \{1, 2, 3, 4, 5\}$$
$$E = \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{3, 5\}\},$$

then $G = (V, E)$ is a graph.

Definition 6.1.2. *Let $G = (V, E)$ be a graph.*

- *Given an edge $e \in E$, we call the elements of e the* endpoints *of e, and we say that e is* incident *to these vertices.*
- *If $u, w \in V$ and $\{u, w\} \in E$, then we say that u and w are* adjacent *or* linked*.*

We can also view graphs as certain types of relations. Recall that a relation on a set V is a subset of V^2, i.e. a set of ordered pairs, while edges in a graph are *sets* with 2 elements. However, if we consider symmetric relations, then we can just ignore the fact that edges are unordered because whenever we have the ordered pair (u, w) we also have the ordered pair (w, u). We need one other condition as well.

Definition 6.1.3. *A relation R on a set A is* irreflexive *if $(a, a) \notin R$ for all $a \in A$.*

Notice that irreflexive does not mean "not reflexive" (if $(a, a) \in R$ some $a \in A$ and $(a, a) \notin R$ for some other $a \in A$, then the relation is neither reflexive nor irreflexive). From this point of view, a graph can be described as a nonempty set V together with a relation on V that is symmetric and irreflexive. For example, we can interpret the above graph as follows:

$$V = \{1, 2, 3, 4, 5\}$$
$$R = \{(1, 2), (2, 1), (1, 3), (3, 1), (1, 5), (5, 1), (3, 5), (5, 3)\}$$

Definition 6.1.4. *Let G be a finite graph with n vertices $v_1, v_2, \ldots, v_n$ listed in some order. We define an $n \times n$ matrix A called the* adjacency matrix *of G by letting*

$$a_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

For the above example with vertices listed in order $1, 2, 3, 4, 5$, we have the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If we change the order of the vertices to be $4, 3, 5, 2, 1$, then we have the adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Notice that the adjacency matrix A of any finite graph is symmetric (i.e. $a_{i,j} = a_{j,i}$ for all i and j, or alternatively A is equal to its transpose), has each entry equal to either 0 or 1, and has all diagonal entries equal to 0. Furthermore, it's not hard to see that every matrix with these properties arises as the adjacency matrix of some finite graph.

Definition 6.1.5. *Let G be a finite graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$ each listed in some order. We define an $n \times m$ matrix B called the* incidence matrix *of G by letting*

$$b_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is an endpoint of } e_j \\ 0 & \text{otherwise} \end{cases}$$

For the above example with vertices listed in order $1, 2, 3, 4, 5$ and edges as $\{1, 2\}, \{1, 3\}, \{1, 5\}, \{3, 5\}$, we have the incidence matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Notice that incidence matrix has each entry equal to either 0 or 1, and has exactly two 1's in each column. Furthermore, it's not hard to see that every matrix with these properties arises as the incidence matrix of some graph.

Definition 6.1.6. *We define the following graphs.*

- *For each $n \in \mathbb{N}^+$, let K_n be the graph with vertex set $V = [n]$ and edge set*

$$E = \{\{i, j\} : 1 \leq i \leq n, 1 \leq j \leq n, \text{ and } i \neq j\}$$

equal to the set of all subsets of V of cardinality 2, i.e. every pair of distinct vertices are linked. We call K_n the complete graph *on n vertices.*

- *For each $n \in \mathbb{N}^+$, let P_n be the graph with vertex set $V = [n]$ and edge set*

$$E = \{\{i, i+1\} : 1 \leq i \leq n-1\}.$$

  We call P_n the path graph *on n vertices.*

- *For each $n \in \mathbb{N}^+$ with $n \geq 3$, let C_n be the graph with vertex set $V = [n]$ and edge set*

$$E = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}.$$

  We call C_n the cycle graph *on n vertices.*

- *For each $m, n \in \mathbb{N}^+$, let $K_{m,n}$ be the graph with vertex set $V = [m+n]$ and edge set*

$$E = \{\{i, j\} : 1 \leq i \leq m \text{ and } m+1 \leq j \leq n\}.$$

  We call $K_{m,n}$ a complete bipartite graph.

In our definition of a graph, given any two vertices, either they are linked by one edge or they are not. Also, in a graph, edges always have two distinct endpoints, so there are no "loops". By relaxing these conditions, we define a broader class of objects that we'll call multigraphs. Since two distinct vertices can have many different edges linking them in this context, the definition is more involved.

Definition 6.1.7. *A* multigraph *G is a triple (V, E, f) of sets such that:*

- *V is a nonempty set.*
- *E is a set disjoint from V.*
- *f is a function with domain E such that $f(e)$ is a subset of V of cardinality either 1 or 2 for each $e \in E$. In other words, $f \colon E \to \mathcal{P}_1(V) \cup \mathcal{P}_2(V)$.*

Elements of V are called vertices*, and elements of E are called* edges*. We say that G is finite if both V and E are finite (notice that it possible that V is finite but E is infinite).*

Definition 6.1.8. *Let $G = (V, E, f)$ be a multigraph.*

- *Given an edge $e \in E$, we call the elements of $f(e)$ the* endpoints *of e, and we say that e is* incident *to these vertices.*
- *If $u, w \in V$ and there is an edge $e \in E$ with $f(e) = \{u, w\}$, then we say that u and w are* adjacent *or* linked*.*
- *We call an edge $e \in E$ a* loop *if $f(e)$ has only 1 element (i.e. if e has only 1 endpoint).*

For example, let

$$V = \{1, 2, 3\}$$
$$E = \{a, b, c, d, e\}$$

and define f by letting:

- $f(a) = \{1, 2\}$.
- $f(b) = \{1, 3\}$
- $f(c) = \{3\}$

- $f(d) = \{1, 2\}$
- $f(e) = \{1, 2\}$

We then have that $G = (V, E, f)$ is a graph. Intuitively, we can think of this graph as follows. There are three vertices labeled 1, 2, and 3. We have one edge linking vertices 1 and 3, three edges linking vertices 1 and 2, and one edge that is a loop at vertex 3 (so both of its endpoints are vertex 3).

Although the definitions of graphs and multigraphs are fundamentally different, we can interpret every graph as a multigraph. For example, recall our graph

$$V = \{1, 2, 3, 4, 5\}$$
$$E = \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{3, 5\}\},$$

We can interpret this as a mutligraph by letting keeping V, letting $E' = \{e_1, e_2, e_3, e_4\}$, and define f by

- $f(e_1) = \{1, 2\}$
- $f(e_2) = \{1, 3\}$
- $f(e_3) = \{1, 5\}$
- $f(e_4) = \{3, 5\}$

Alternatively, and much more simply, we can keep both V and E, and just let $f \colon E \to \mathcal{P}_2(V)$ be the function where $f(e) = e$ (since, after all, in a graph an element of E is a subset of V of cardinality 2). This always works as long as V and E are disjoint.

One can define analogues of the adjacency and incidence matrices for multigraphs, but there there is not a standard way to deal with loops.

- If G is a finite multigraph, and we've listed the vertices in order as $v_1, v_2, \ldots, v_n$, then we define the $n \times n$ adjacency matrix A as follows. If $i \neq j$, let $a_{i,j}$ be the number of edges with endpoints v_i and v_j. For the diagonal entries, it is natural to let $a_{i,i}$ be the number of loops at v_i, but there is also a strong argument for letting $a_{i,i}$ be twice the number of loops at v_i (so we're counting the endpoint as having "multiplicity" 2).

- If G is a finite multigraph, and we've listed the vertices and edges in order as $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$, then we define the $n \times m$ incidence matrix B as follows. If e_j is not a loop, let

$$b_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is an endpoint of } e_j \\ 0 & \text{otherwise} \end{cases}$$

  as in the case for graphs. If e_j is a loop with single endpoint v_i, then we define $b_{k,j} = 0$ for all $k \neq i$, and we let $b_{i,j}$ equal either 1 or 2, depending on our preference (as in the adjacency matrix case).

Since we won't be dealing with adjacency and incidence matrices of multigraphs very often, we'll just stipulate which version we are using

Definition 6.1.9. *Let G be a multigraph and let $v \in V$. The* degree *of v, denoted by $d(v)$, is the number of edges incident to v, where each loop incident to v is counted twice.*

Proposition 6.1.10. *If G is a finite multigraph with m edges, then*

$$\sum_{v \in V} d(v) = 2m$$

Proof. Every edge has two endpoints (if we count the unique endpoint of any loop twice), so contributes 2 to the sum on the left hand-side. The result follows. □

Corollary 6.1.11. *A finite multigraph has an even number of vertices of odd degree.*

Proof. Let G be a finite multigraph with m edges. Proposition 6.1.10 tells us that

$$\sum_{v \in V} d(v) = 2m,$$

so

$$\sum_{v \in V} d(v)$$

is even. If there were odd number of vertices of odd degree, then this sum would be odd, which is a contradiction. □

Definition 6.1.12. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. We say that H is a* subgraph *of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.*

Definition 6.1.13. *Let $G = (V, E)$ be a graph.*

- *For any subset $F \subseteq E$, we let $G - F$ be the subgraph of G with vertex set equal to V and edge set equal to $E \backslash F$. If $F = \{e\}$, we write $G - e$ instead of $G - \{e\}$.*
- *For any subset $U \subseteq V$ with $U \neq V$, we let $G - U$ be the subgraph of G with vertex set $V \backslash U$ and edge set equal to $\{e \in E :$ Both endpoints of e are elements of $V \backslash U\}$. If $U = \{u\}$, we write $G - u$ instead of $G - \{u\}$.*
- *For any subset $U \subseteq V$ with $U \neq \emptyset$, we let $G[U]$ be the subgraph of G with vertex set U and edge set equal to $\{e \in E :$ Both endpoints of e are elements of $U\}$. Notice that $G[U] = G - (V/U)$. We call $G[U]$ the subgraph of G* induced *by U.*

Thus, $G - F$ is obtained by deleting all of the edges in F, while $G - U$ is obtained by deleting all of the vertices in U *as well as* all edges incident to some vertex in U. Intuitively, an induced subgraph of G is one obtained by only deleting vertices (and all of their associated edges), whereas a general subgraph of G is one obtained by deleting vertices (and all of their associated edges) along with possibly deleting additional edges whose endpoints are still alive. There can exist subgraphs of a graph G that are not induced subgraphs, such as the result of deleting one edge.

Definition 6.1.14. *Let $G = (V_G, E_G, f_G)$ and $H = (V_H, E_H, f_H)$ be multigraphs. We say that H is a* submultigraph *of G if $V_H \subseteq V_G$, $E_H \subseteq E_G$, and f_H is the restriction of f_G to the set E_H.*

One can similarly define $G - F$, $G - U$, and $G[U]$ for multigraphs.

Definition 6.1.15. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An* isomorphism *from G_1 to G_2 is a bijection $g \colon V_1 \to V_2$ such that for all $u, w \in V_1$, we have that $\{u, w\} \in E_1$ if and only if $\{g(u), g(w)\} \in E_2$.*

For example, consider the following two graphs. Let G_1 be the graph where

$$V_1 = \{1, 2, 3, 4, 5\}$$
$$E_1 = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$$

and let G_2 be the graph where

$$V_2 = \{a, b, c, d, e\}$$
$$E_2 = \{\{a, c\}, \{a, d\}, \{a, e\}, \{b, e\}, \{c, d\}\}$$

We then have that the function $g\colon V_1 \to V_2$ defined by

$$\begin{aligned} g(1) &= e \\ g(2) &= b \\ g(3) &= a \\ g(4) &= c \\ g(5) &= d \end{aligned}$$

is an isomorphism.

Definition 6.1.16. *Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we say that G_1 is* isomorphic *to G_2, and write $G_1 \cong G_2$, if there exists an isomorphism from G_1 to G_2.*

Intuitively, two graphs G_1 and G_2 are isomorphic when it is possible to relabel the names of the vertices so that the graphs look the same, and the function g is precisely the "translation" between the names of the vertices on each side. In terms of pictures, saying that $G_1 \cong G_2$ is the same as saying that it is possible to draw the graphs in identical fashions, provided we can place the vertices anywhere that we would like.

Definition 6.1.17. *Let $G_1 = (V_1, E_1, f_1)$ and $G_2 = (V_2, E_2, f_2)$ be multigraphs. An* isomorphism *is a pair of bijections $g\colon V_1 \to V_2$ and $h\colon E_1 \to E_2$ such that for all $e \in E_1$, if $f_1(e) = \{u, w\}$, then $f_2(h(e)) = \{g(u), g(w)\}$, and if $f_1(e) = \{v\}$, then $f_2(h(e)) = \{g(v)\}$*

6.2 Walks, Paths, Cycles, and Connected Components

Definition 6.2.1. *Let $G = (V, E, f)$ be a multigraph.*

- *A* walk *in G is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ where each $v_i \in V$, each $e_i \in E$, and where $f(e_i) = \{v_{i-1}, v_i\}$ for all i (i.e. the endpoints of e_i are v_{i-1} and v_i). We allow walks to consists of a single vertex v_0 and no edges.*
- *A* trail *in G is a walk with no repeated edges, i.e. where $e_i \neq e_j$ whenever $i \neq j$.*
- *A* path *in G is a walk with no repeated vertices, i.e. where $v_i \neq v_j$ whenever $i \neq j$.*
- *A* closed walk *in G is a walk is a walk where $v_0 = v_k$. Similarly, a* closed trail *is a trail where $v_0 = v_k$.*
- *A u, w-walk in G is a walk with $v_0 = u$ and $v_k = w$ (similarly for a u, w-trail and a u, w-path).*

Definition 6.2.2. *Let G be a multigraph. Given a walk in G, we define the length of the walk to be the number of edges it contains, counting repetition. In other words, the length of the walk*

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

is k (which is one less than the number of vertices).

Proposition 6.2.3. *Let G be a multigraph.*

- *Every path in G is a trail.*
- *Every trail in G is a walk.*

Proof. Clearly every trail in G is a walk because a trail is by definition a walk. We need to prove that every path in G is a trail. Suppose then that

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

is a path in G. We need to show that this is a trail, so suppose for the sake of obtaining a contradiction that $e_i = e_j$ where $i < j$. Since $e_i = e_j$, we have that e_i and e_j have the same endpoints, which is to say that $\{v_{i-1}, v_i\} = \{v_{j-1}, v_j\}$. It follows that either $v_{i-1} = v_{j-1}$ or $v_{i-1} = v_j$. Since $i - 1 < i \leq j - 1$, in either case we have violated the definition of a path because we have found a repeated vertex. This is a contradiction, so our path must be a trail. □

Proposition 6.2.4. *Let G be a multigraph and let $u, w \in G$. The following are equivalent.*

1. *There is a u, w-walk in G.*
2. *There is a u, w-trail in G.*
3. *There is a u, w-path in G.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate from the previous proposition. We now prove that (1) $\Rightarrow$ (3). Suppose that there is a u, w-walk in G. Fix a u, w-walk in G of shortest possible length, say it is:

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

where $v_0 = u$ and $v_k = w$. We argue that this walk is a u, w-path. Suppose for the sake of obtaining a contradiction that some vertex is repeated, say $v_i = v_j$ where $i < j$. We then have that

$$v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, e_{j+1}, v_{j+1}, \ldots v_{k-1}, e_k, v_k$$

is a u, w-walk because $v_i = v_j$ (so the set of endpoints of e_{j+1} equals $\{v_j, v_{j+1}\} = \{v_i, v_{j+1}\}$). Furthermore, this walk has length

$$i + (k - j) = k - (j - i) < k$$

Thus, we have produced a u, w-walk in G of shorter length, which is a contradiction. It follows that our above u, w-walk is in fact a u, w-path in G. □

Proposition 6.2.5. *Let G be a multigraph. Define a relation $\sim$ on V by letting $u \sim w$ mean that there is a u, w-walk in G. We then have that $\sim$ is an equivalence relation.*

Proof. We check the properties.

- Reflexive: For any $u \in V$, we have that the single vertex u is a u, u-walk in G, so $u \sim u$.
- Symmetric: Suppose that $u \sim w$. Fix a u, w-walk, say it is:

  $$u = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = w$$

  We then have that

  $$w = v_k, e_k, v_{k-1}, \ldots, v_2, e_2, v_1, e_1, v_0 = u$$

  is a w, u-walk in G, so $w \sim u$.

- Suppose that $u \sim w$ and $w \sim y$. Fix a u, w-walk

$$u = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = w$$

and a w, y-walk

$$w = x_0, f_1, x_1, f_2, x_2, \ldots, x_{\ell-1}, f_\ell, x_\ell = y$$

We then have that

$$u = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = w = x_0, f_1, x_1, \ldots, x_{\ell-1}, f_\ell, x_\ell = y$$

is a u, y-walk in G, so $u \sim y$.

□

Definition 6.2.6. *Let G be a multigraph and let $\sim$ be the above relation on V. We know from our general theory of equivalence relations that the equivalence classes of $\sim$ are subsets of V that partition V. A* connecting component *of G is a subgraph of G of the form $G[U]$ for some equivalence class U of $\sim$.*

We know that each vertex of a multigraph G appears in a unique connected component of G because the equivalence classes of $\sim$ partition V. We now show that the same is true for edges.

Proposition 6.2.7. *Let G be a multigraph. Every edge of G appears in a unique connected component of G.*

Proof. Let $e \in E$ be arbitrary. Let u and w be the endpoints of E. Since u, e, w is a u, w-walk in G, we have $u \sim w$. Thus, if we let $U = \overline{u}$ be the equivalence class of u, then $u, w \in U$, and hence $e \in G[U]$. Furthermore, since the equivalence classes partition V, the vertices u and w are not in any other equivalence class, and hence e is not an element of any other connected component. □

Definition 6.2.8. *A multigraph G is* connected *if it has one connected component. In other words, G is a connected if there exists a u, w-walk in G for all $u, w \in V$.*

Proposition 6.2.9. *If G is a multigraph, then every connected component of G is a connected graph.*

Proof. Let G be a multigraph, and let $U \subseteq V$ be an equivalence class of $\sim$. Let $u, w \in U$ be arbitrary. Since u and w are elements of the same equivalence class, we have $u \sim w$, and hence we can fix a u, w-walk

$$u = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = w$$

in G. Notice that for each i with $1 \leq i \leq k$, we have that

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{i-1}, e_i, v_i$$

is a walk in G, so $u \sim v_i$. It follows that $v_i \in U$ for all i with $1 \leq i \leq k$. Now given any i with $2 \leq i \leq k$, we have that both $v_{i-1} \in U$ and $v_i \in U$, so e_i is an edge of $G[U]$. It follows that the walk

$$u = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = w$$

is also a u, w-walk in $G[U]$. We have shown that for any two vertices $u, w \in U$, there is a u, w-walk in $G[U]$, so $G[U]$ is connected. □

Proposition 6.2.10. *Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and let A be the adjacency matrix. For all $k \in \mathbb{N}^+$, the (i, j) entry of the matrix A^k equals the number of v_i, v_j-walks in G of length k.*

Proof. We prove the result by induction on k.

- *Base Case:* Since
$$a_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$
and a walk of length 1 consists of a single edge, it follows that $a_{i,j}$ is the number of v_i, v_j-walks of length 1 in G.

- *Inductive Step:* Suppose then that the result is true for k. Letting $B = A^k$, we then have that $b_{i,j}$ is the number of v_i, v_j-walks of length k in G. Let $C = A^{k+1} = A^k A = BA$. Fix i and j, and let
$$L = \{\ell \in [n] : v_\ell \text{ is adjacent to } v_j\}$$
We have
$$\begin{aligned} c_{i,j} &= \sum_{\ell=1}^{n} b_{i,\ell} a_{\ell,j} \\ &= \sum_{\ell \in L} b_{i,\ell} a_{\ell,j} && (\text{since } a_{\ell,j} = 0 \text{ if } \ell \notin L) \\ &= \sum_{\ell \in L} b_{i,\ell} && (\text{since } a_{\ell,j} = 1 \text{ if } \ell \in L) \end{aligned}$$
Now given any v_i, v_j-walk of length $k+1$, the second to last vertex in the sequence must be a vertex adjacent to v_j, hence must equal v_ℓ for some $\ell \in L$. Thus, since G is a graph (i.e. not a multigraph), a v_i, v_j-walk of length $k+1$ is completely and uniquely determined by choice of v_ℓ for some $\ell \in L$ as the second to last vertex, together with a v_i, v_ℓ walk of length k. By induction, adding up the number of such walks amounts to calculating the last sum above. Therefore, $c_{i,j}$ is the is the number of v_i, v_j-walks of length $k+1$ in G. The result follows by induction.

□

Above, we defined the cycle graphs C_n for each $n \in \mathbb{N}^+$ with $n \geq 3$ as follows. Given $n \in \mathbb{N}^+$ with $n \geq 3$, we let C_n be the graph with vertex set $V = [n]$ and edge set
$$E = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}.$$
For $n = 1$ and $n = 2$, we can also define *multigraphs* C_1 and C_2 as follows.

- C_1 is the multigraph with vertex set $[1] = \{1\}$ and one edge that is a loop at 1.
- C_2 is the multigraph with vertex set $[2] = \{1, 2\}$ and two edges, each of which have endpoints 1 and 2 (so there is one double edge).

Together together, the C_n for $n \in \mathbb{N}^+$ form the cycle (multi)graphs. We now define cycles *within* graphs.

Definition 6.2.11. *Let G be a multigraph. A* cycle *in G is a submultigraph of G that is isomorphic to C_n for some $n \in \mathbb{N}^+$.*

Although cycles are certain subgraphs of G, we often find them by finding closed walks without repeated vertices or edges.

Proposition 6.2.12. *Let G be a multigraph.*

1. *Suppose that $k \geq 1$ and that*

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

is a closed walk in G without repeated edges and without repeated vertices other than $v_0 = v_k$ (i.e. where $e_i \neq e_j$ whenever $1 \leq i < j < k$ and $v_i \neq v_j$ whenever $0 \leq i < j < k$). If we let $U = \{v_0, v_1, \ldots, v_{k-1}\}$ and $F = \{e_1, e_2, \ldots, e_k\}$, then $H = (U, F)$ is a submultigraph of G that is isomorphic to C_k, so $H = (U, F)$ is a cycle.

2. *Conversely, suppose that $H = (U, F)$ is a cycle of G. It is then possible to list the vertices of U as $v_0, v_1, \ldots, v_{k-1}$ and list the edges of F as $e_1, e_2, \ldots, e_k$ in such a way that*

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

is a closed walk without repeated edges and without repeated vertices other than $v_0 = v_k$ (i.e. where $e_i \neq e_j$ whenever $1 \leq i < j < k$ and $v_i \neq v_j$ whenever $0 \leq i < j < k$).

Proof. 1. We prove the result when $k \geq 3$ (the cases for $k = 1$ and $k = 2$ are similar, but there we need to treat C_k as a multigraph). Let

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

be a closed walk in G without repeated edges and without repeated vertices other than $v_0 = v_k$. Define a function $g\colon V_{C_k} \to U$ by letting $g(i) = v_i$ for all i. Also, define a function $h\colon E_{C_k} \to F$ by letting $h(\{i, i+1\}) = e_i$ for all i with $1 \leq i \leq k-1$, and letting $h(\{1, k\}) = e_k$. We then have that g and h are bijective because the v_i and e_i are distinct. These functions give an isomorphism of $H = (U, F)$ with C_k, so H is a cycle.

2. Suppose that $H = (U, F)$ is a cycle of G. Fix bijections $g\colon V_{C_k} \to U$ and $h\colon E_{C_k} \to F$ that form an isomorphism. Let $v_i = g(i)$ for all i with $1 \leq i \leq k$, and let $v_0 = g(k)$. Also, let $e_i = h(\{i, i+1\})$ for all i with $1 \leq i < k$, and let $e_k = h(\{1, k\})$. Since g and h are bijections, it follows that the e_i are distinct and the v_i are distinct, other than $v_0 = v_k$. Furthermore, since g and h form an isomorphism, we have that

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

is a closed walk. This completes the proof.

□

Since walks are easier to understand and work with than isomorphisms, one may ask why we do not define cycles as these closed walks. The answer is that certain distinct closed walks give the "same" cycle. For example, consider the graph $G = (V, E)$ where:

$$V = \{1, 2, 3, 4\}$$
$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}$$

Although this graph only has one cycle, the following three closed walks are all distinct even though the "trace" the same cycle:

- $1, \{1, 2\}, 2, \{2, 3\}, 3, \{1, 3\}, 1$
- $2, \{2, 3\}, 3, \{1, 3\}, 1, \{1, 2\}, 2$
- $3, \{2, 3\}, 2, \{1, 2\}, 1, \{1, 3\}, 3$

There are 3 other possible such closed walks as well! Thus, if we want to *count* cycles, then our definition is superior.

The previous proposition lets us view cycles as arising from closed walks without repeated edges or vertices. One may ask whether we need this restriction. We certainly do not want to allow edges to repeat, because if we retrace our steps then that should not be a cycle (and the result will not be isomorphic to C_n). Simply saying that the edges do not repeat is also not enough. For example, consider the graph $G = (V, E)$ where:

$$V = \{1, 2, 3, 4, 5\}$$
$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$$

This graph looks like a bow tie. If we follow the natural walk around this bow tie shape, then we do not repeat edges, but we do repeat the vertex 3 in the middle (in addition to the starting/ending vertex 1). This graph is not isomorphic to C_5, so we do not count it as a cycle.

How about if we only enforce that there are no repeated vertices? In trivial cases, this is not enough. For example, if G is a graph, and $e \in E$ is an edge with distinct endpoints u and w, then u, e, w, e, u is a closed walk without repeated vertices (other than the beginning/end), but it is not a cycle. However, for closed walks of length $k \geq 3$, having no repeated vertices automatically gives that there are no repeated edges.

Proposition 6.2.13. *Let G be a multigraph. Suppose that $k \geq 3$ and that*

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

is a closed walk without any repeated vertices other than $v_0 = v_k$. We then have that $e_i \neq e_j$ whenever $1 \leq i < j \leq k$. Thus, if we let $U = \{v_0, v_1, \ldots, v_{k-1}\}$ and $F = \{e_1, e_2, \ldots, e_k\}$, then $H = (U, F)$ is a submultigraph of G that is isomorphic to C_k, so $H = (U, F)$ is a cycle.

Proof. Notice that

$$v_0, e_1, v_1, \ldots, e_{k-1}, v_{k-1}$$

is a path in G, so it is a trail in G by Proposition 6.2.3. Hence $e_i \neq e_j$ whenever $1 \leq i < j \leq k - 1$. We also have that

$$v_1, e_2, \ldots, v_{k-1}, e_k, v_k$$

is a path in G, hence a trail, and so $e_i \neq e_j$ whenever $2 \leq i < j \leq k$. Finally, notice that $e_1 \neq e_k$ because v_1 is an endpoint of e_1, v_{k-1} is an endpoint of e_k, and $v_1 \neq v_{k-1}$ because $k - 1 \geq 2$ (as $k \geq 3$). The last statement now follows from Proposition 6.2.12. □

Despite the fact that a closed walk of length at least 1 without repeated edges, i.e. a closed trail of length at least 1, need not be a cycle (as in our bow tie example), it turns out that if G contains such a closed trail, then G also contains a cycle.

Proposition 6.2.14. *If G contains a closed trail of length at least 1, then G contains a cycle.*

Proof. Fix a shortest possible closed trail of length at least 1, say it is:

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

Notice that if $k = 1$, then we have the closed trail v_0, e_1, v_0, which is a cycle (it is a loop and is isomorphic to C_1), so we are done. Suppose then that $k \geq 2$. We claim that the vertices in the list $v_0, v_1, \ldots, v_{k-1}$ are all distinct. Suppose, for the sake of obtaining a contradiction, that this is not true. Fix i and j with $0 \leq i < j \leq k - 1$ such that $v_i = v_j$. We then have that

$$v_i, e_{i+1}, v_{i+1}, \ldots, v_{j-1}, e_j, v_j$$

is a closed trail of length $j-i$. Since $1 \leq j-i \leq k-1$, this would give an example of a nontrivial closed trail of length strictly less than k, which is a contradiction. Thus, the vertices in the list $v_0, v_1, \ldots, v_{k-1}$ are all distinct. Furthermore, since $v_0 = v_k$, we have that $v_i \neq v_k$ whenever $1 \leq i \leq k-1$. Therefore, the vertices in our closed trail

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

are all distinct (except for $v_0 = v_k$). Using Proposition 6.2.12, we conclude that G contains a cycle. □

Proposition 6.2.15. *Let G be a multigraph with the following properties:*

1. *V is finite.*
2. *$E \neq \emptyset$.*
3. *$d(v) \neq 1$ for all $v \in V$*

We then have that G contains a cycle.

Proof. If G has any loops or multiple edges, then it trivially has a cycle. Suppose then that G is a graph. Since V is finite, any path must have length at most $|V|$, and hence we may fix a longest possible path in G:

$$v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

Since this is a path, no vertex is repeated, and so no edge is repeated either (because paths are trails). We also have $k \geq 1$ since G has at least one edge, and this edge is not a loop. Since e_1 has v_0 as an endpoint, we conclude that $d(v_0) \geq 1$. Now e_1 is not a loop, so there must be an edge $f \neq e_1$ such that f is incident to v_0. As f is not a loop, we know that f is incident to a vertex other than v_0. Let w be the other endpoint of f, so $w \neq v_0$. Now we must have that $w = v_i$ for some i with $0 \leq i \leq k$, because if $w \neq v_i$ for all i with $0 \leq i \leq k$, then

$$w, f, v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$$

would be a longer path in G, contradicting our choice of a longest possible path in G. Thus, we can fix ℓ with $0 \leq \ell \leq k$ such that $w = v_\ell$. Notice that $\ell \neq 0$ because $w \neq v_0$. Also, $w \neq v_1$ because $f \neq e_1$ and we are assuming that G does not have multiple edges. Thus, we must have that $\ell \geq 2$. Now since the v_i are distinct, we know that that $w \neq v_i$ whenever $0 \leq i < \ell$. Therefore,

$$w, f, v_0, e_1, v_1, \ldots, v_{\ell-1}, e_\ell, v_\ell$$

is a closed walk in G without repeated vertices (other than $w = v_\ell$). Furthermore, since $\ell \geq 2$, this closed walk has length at least 3, we may use Proposition 6.2.13 to conclude that G contains a cycle. □

Proposition 6.2.16. *Let G be a connected multigraph and let e be an edge of G. The following are equivalent.*

1. *$G - e$ is connected.*
2. *G has a cycle containing e.*

Proof. (1) ⇒ (2): Suppose that $G - e$ is connected. First notice that if e is a loop, then G certainly has a cycle containing e. Suppose then that e is not a loop. Let the endpoints of e be u and w. Since $G - e$ is connected, we can fix a u, w-path

$$u = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = w$$

in the graph $G - e$. We then have that

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k, e, v_0$$

is a closed walk in G without repeated edges of vertices. Using Proposition 6.2.12, it follows that G has a cycle containing e.

(2) $\Rightarrow$ (1): Suppose that C is a cycle of G containing e. Let x and y be the endpoints of e (it is possible that $x = y$ if e is a loop). Let $\sim$ denote the connectivity relation in $G - e$. Now let $u, w \in V$ be arbitrary. We will show that $u \sim w$. Since G is connected, we may fix a u, w-path in G, say

$$u = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = w.$$

Now if $e \neq e_i$ for all i, then this u, w-path exists in $G - e$, and we have $u \sim w$. Suppose instead that $e = e_i$ for some i. Since this is a path, it is also a trail by Proposition 6.2.3, and hence there is a unique ℓ with $1 \leq \ell \leq k$ such that $e = e_\ell$. Now the endpoints of e are x and y, so either $x = v_{\ell-1}$ and $y = v_\ell$, or $x = v_\ell$ and $y = v_{\ell-1}$. We consider these two cases:

- Suppose first that $x = v_{\ell-1}$ and $y = v_\ell$. Since $e \neq e_i$ whenever $1 \leq i \leq \ell - 1$, we have that

  $$u = v_0, e_1, v_1, \ldots, e_{\ell-1}, v_{\ell-1} = x$$

  is a u, x-walk in $G - e$, so $u \sim x$. Similarly, since $e \neq e_i$ whenever $\ell + 1 \leq i \leq k$, we have that

  $$y = v_{\ell+1}, e_{\ell+2}, v_{\ell+2}, \ldots, e_k, v_k = w$$

  is a y, w-walk in $G - e$, so $y \sim x$. Finally, since e lies in a cycle of G containing e, we see that $x \sim y$ by following such a cycle around in the other direction. Combining $u \sim x$, $y \sim w$, and $x \sim y$ with the fact that $\sim$ is an equivalence relation, we conclude that $u \sim w$. Thus, there is a u, w-walk in $G - e$.

- Suppose now that $x = v_\ell$ and $y = v_{\ell-1}$. Arguing as in the previous case, we have that $u \sim y$, $x \sim w$, and $y \sim x$, so $u \sim w$. Thus, there is a u, w-walk in $G - e$.

We have show that there is a u, w-walk in $G - e$ for all vertices u and w, so $G - e$ is connected. □

6.3 Trees and Forests

Definition 6.3.1. *A* tree *is a connected acyclic graph.*

Definition 6.3.2. *A* forest *is an acyclic graph.*

Although we've defined a trees and forests as certain types of *graphs*, notice that we can also define it as a connected acyclic multigraph because the lack of cycles rules out loops and multiple edges.

Proposition 6.3.3. *If G is a forest, then every connected component of G is a tree.*

Proof. Let H be a connected component of G. We know that H is connected by Proposition 6.2.9. Furthermore, H is acyclic, because a cycle in H would be a cycle in G. Therefore, H is a tree. □

Proposition 6.3.4. *If T is a tree with at least 2 vertices, then no vertex of T is isolated, i.e. no vertex of T has degree 0.*

Proof. Let $v \in V$ be arbitrary. Since T has at least 2 vertices, we can fix $w \in V$ with $w \neq v$. Now T is connected, so there exists a v, w-path in T. The first edge of this path must be incident to v, so $d(v) \geq 1$. □

Definition 6.3.5. *Let T be a tree. A* leaf *of T is a vertex of degree 1.*

Proposition 6.3.6. *If T is a finite tree with $n \geq 2$ vertices, then T has a leaf.*

Proof. Let T be a finite tree with $n \geq 2$ many vertices. Since T is connected and $n \geq 2$, we must have that $E \neq \emptyset$ by Proposition 6.3.4. Now if $d(v) \neq 1$ for all $v \in V$, then Proposition 6.2.15 would imply that T contains a cycle, which is a contradiction. Therefore, there must exist $v \in V$ with $d(v) = 1$, and such a v is a leaf. □

Proposition 6.3.7. *If v is a leaf of a tree T, then $T - v$ is a tree.*

Proof. Let T be a tree, and let v be a leaf of T. Let e be the unique edge incident to v. Since $T - v$ is a subgraph of T, it follows that $T - v$ is acyclic (a cycle in $T - v$ would be a cycle in T). Let u and w be arbitrary vertices of $T - v$. Since T is connected, we can fix a u, w-path in T. Notice that e and v can not occur on this path (because e is the the only edge incident to v, and so such a purported path would need to use e twice). Therefore, there is a u, w-path in $T - v$. Thus, $T - v$ is connected. □

The above two results allow one to prove results about finite trees by induction on the number of vertices. This is an extremely powerful tool.

Theorem 6.3.8. *If T is a tree with n vertices, then T has exactly $n - 1$ edges.*

Proof. By induction on n, i.e. we prove the statement that "Every tree on n vertices has exactly $n-1$ edges" by induction. If $n = 1$, this is trivial. Suppose that $n \in \mathbb{N}^+$ and we know the result for n. Let T be a tree on $n + 1$ vertices. Fix a leaf v and the unique edge e incident to it. We then have that $T - v$ is a tree on n vertices, so has $n - 1$ edges by induction. Since T has one more edge that $T - v$, we conclude that T has $(n - 1) + 1 = n = (n + 1) - 1$ many edges. The result follows. □

Corollary 6.3.9. *If T is a finite tree with $n \geq 2$ vertices, then T has at least two leaves.*

Proof. Let T be a finite tree with $n \geq 2$ vertices. By Theorem 6.3.8, we know that T has exactly $n-1$ edges. Thus

$$\sum_{v \in V} d(v) = 2(n - 1) = 2n - 2.$$

Now if there is only one leaf, then $d(v) \geq 2$ for all other vertices v by Proposition 6.3.4, so

$$\begin{aligned} \sum_{v \in V} d(v) &\geq 1 + 2(n - 1) \\ &= 2n - 1 \\ &> 2n - 2, \end{aligned}$$

a contradiction. Thus, T must have at least two leaves. □

Proposition 6.3.10. *If G is a finite forest with n vertices and k connected components, then G has $n - k$ edges.*

Proof. Let the components of G be $H_1, H_2, \ldots, H_k$. Suppose that H_i has m_i many vertices for each i. Since each vertex lies in a unique connected component (recall that the vertices of a connected component are equivalence classes of $\sim$), we have

$$\sum_{i=1}^{k} m_i = n.$$

Now each H_i is a tree by Proposition 6.3.3, so we know that H_i has $m_i - 1$ many edges for each i by Theorem 6.3.8. Since every edge is in a unique H_i by Proposition 6.2.7, it follows that the number of edges in G equals

$$\begin{aligned} \sum_{i=1}^{k} (m_i - 1) &= \left(\sum_{i=1}^{k} m_i \right) - k \\ &= n - k. \end{aligned}$$

This completes the proof. □

Definition 6.3.11. *Let G be a connected graph. A* spanning tree *of G is a subgraph T of G such that:*

- *$V_T = V_G$ (i.e. T is obtained from G by only deleting edges).*
- *T is a tree.*

Proposition 6.3.12. *Every finite connected graph has a spanning tree.*

Intuitively, we can argue this as follows. Suppose that G is a finite connected graph. If G has no cycles, then G itself is a spanning tree of G, and we are done. If G contains a cycle, then we pick an arbitrary edge e in a cycle, and notice that $G - e$ is a connected subgraph (by Proposition 6.2.16) with one fewer edge. If this subgraph of G has no cycles, then it is a spanning tree of G. Otherwise, if we remove another edge from a cycle in $G-e$, then the result is a connected subgraph with two fewer edges than G. Continue this process, and notice that we must stop because G has only finitely many edges. More formally, we can bypass this "continue" argument in the following way.

Proof. Let G be a finite connected graph. Notice that there is at least one connected subgraph H of G with $V_H = V_G$, namely G itself. Amongst all such connected subgraphs H of G with $V_H = V_G$, choose one with the least possible number of edges, and call the resulting subgraph T. We then have that $V_T = V_G$ and that T is connected by definition. If T contained a cycle, then we could fix an arbitrary edge e in such a cycle, and notice that $T - e$ is a connected subgraph of G (by Proposition 6.2.16) with one fewer edge, a contradiction. Therefore, T is acyclic as well. It follows that T is a tree, and hence a spanning tree of G. □

Corollary 6.3.13. *If G is a finite connected graph with n vertices, then G contains at least $n - 1$ edges.*

Proof. Fix a spanning tree T of G. We then have that T contains $n - 1$ edges, so G contains at least $n - 1$ edges. □

Theorem 6.3.14. *Let G be a finite graph with n vertices. The following are equivalent.*

1. *G is a tree.*
2. *G is a connected graph with $n - 1$ edges.*
3. *G is an acyclic graph with $n - 1$ edges.*
4. *G is connected, but $G - e$ is disconnected for every edge e.*
5. *G is acyclic, but $G + e$ has a cycle for any new edge e having both endpoints in V_G.*

Proof.

- (1) ⇒ (2): Immediate from the definition of a tree and Theorem 6.3.8.
- (1) ⇒ (3): Immediate from the definition of a tree and Theorem 6.3.8.
- (1) ⇒ (4): Immediate from the definition of a tree and Proposition 6.2.16.
- (1) ⇒ (5): Suppose that G is a tree, and that e is a new edge. Since $(G + e) - e = G$ is connected, e must be an element of some cycle of $G + e$ by Proposition 6.2.16. In particular, $G + e$ has a cycle.
- (2) ⇒ (4): Immediate from Corollary 6.3.13.
- (4) ⇒ (1): Since $G - e$ is disconnected for every edge e, Proposition 6.2.16 implies that no edge of G is contained in a cycle. Thus, G is acyclic.

- (3) $\Rightarrow$ (1): Since G is acyclic, it is a forest. Let k be the number of connected components of G. By Proposition 6.3.10, we know that G has $n-k$ edges. It follows that $n-k=n-1$, hence $k=1$. Therefore, G is connected, and hence a tree.

- (5) $\Rightarrow$ (1): We need to argue that G is connected. Let e be a new edge with endpoints $u, w \in V$. By assumption, we know that $G+e$ has a cycle, and since G is acyclic it must have a cycle containing e. Fix such a cycle, and walk around it the other way to obtain a u, w-path in G.

□

We now embark on a quest to count the number of trees of a given size. In other words, given $n \in \mathbb{N}^+$, how many trees are there with vertex set $[n]$? Let's consider the case when $n=4$. To determine what we want to count, we first need to deal with a potential source of ambiguity in the question. For example, consider the trees T_1 and T_2 each with vertex set $[4]$, where the edge set of T_1 is

$$\{\{1,2\},\{1,3\},\{1,4\}\}$$

and the edge set of T_2 is

$$\{\{1,2\},\{2,3\},\{2,4\}\}.$$

It is straightforward to check that these are each trees. At first sight, they clearly look different because the edge sets are distinct. However, it's not hard to see that the trees T_1 and T_2 are isomorphic (intuitively, we can draw T_1 by putting 1 in the middle with 3 edges out to to other vertices, and we can draw T_2 by putting 2 in the middle with 3 edges out to the other vertices). Thus, we need to ask whether we are actually counting the number of trees with vertex set $[n]$, or the number is isomorphism types of trees with vertex set $[n]$. Although both questions are interesting, we are going to do the former here. That is, we will consider the above two trees as different.

Suppose now that we want to count the number of trees with vertex set $[4]$. There are 16 such trees, and we give two ways to count this.

1. There are two isomorphism types for a tree with vertex set $[4]$. First, we can have a tree with a vertex of degree 3 that is adjacent to all other vertices (so there are 3 leaves). Second, we can have a tree with exactly 2 leaves and two vertices of degree 2 (recall that a tree on at least 2 vertices has at least 2 leaves, and that the sum of the degrees will be $2\cdot(4-1)=6$), and it's straightforward to check that such a tree is isomorphic to P_4. We now count the number of trees of each isomorphism type.

   - There are exactly 4 trees of the first type, because they are completely determined by the choice of the vertex of degree 3.
   - We now argue that there are exactly 12 trees of the second type. We can choose the two leaves in $\binom{4}{2}=6$ ways. This then determines the vertices of degree 2 (which will be adjacent to each other). To determine the tree, we now need only pick which of these two is adjacent to the smaller leaf, and we have 2 choices. Thus, the number of such trees is $4\cdot 2=12$.

2. Here is another argument. A tree with vertex set $[4]$ will have exactly $4-1=3$ edges. Since there are $\binom{4}{2}=6$ many possible edges, there are $\binom{6}{3}=20$ many graphs with vertex set set $[4]$ having exactly 3 edges. However, some of these will fail to be trees because they are not connected. This happens exactly when the graph consists of a cycle of length 3 and an isolated vertex, and there are 4 such graphs (because there are 4 choices for the isolated vertex). It follows that there are $20=4=16$ many trees with vertex set $[4]$.

Counting the number of trees with vertex sets $[n]$ for other small values of n is also reasonably straightforward:

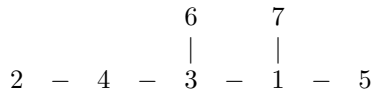
1. There is 1 tree with vertex set $[1]$.

2. There is 1 tree with vertex set [2].
3. There are 3 trees with vertex set [3] (we need only the pick the unique vertex of degree 2).
4. There are 16 trees with vertex set [4] (as seen above).
5. There are 125 trees with vertex set [5], which can be argued through a slightly more complicated analysis than the one for [4].

As we move toward larger values of n, the above method become unwieldy. However, there is a natural pattern in the above numbers, and we will indeed prove the following result using a more sophisticated approach

Theorem 6.3.15 (Cayley's Formula)**.** *For each $n \in \mathbb{N}^+$, the number of trees with vertex set $[n]$ is n^{n-2}.*

In order to prove this, we will "code" each tree by a sequence of numbers of length $n-2$, where each element of the sequence is an integer between 1 and n (inclusive). Given a tree T with vertex set $[n]$, we know that it has $n-1$ edges. Consider the following tree:



The edge set of this tree is

$$\{\{1,3\},\{1,5\},\{1,7\},\{2,4\},\{3,4\},\{3,6\}\}.$$

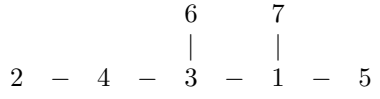
Of course, the edge set is a *set*, so we can reorder it any way we like without affecting the edges, and we can also reorder the two endpoints of an edge. Our first task will be to give an ordering to the edge set in a way that reflects the "structure" of the tree. To do this, given $n \geq 2$ and a tree T on $[n]$, we define two sequences $(a_1, a_2, \ldots, a_{n-1})$ and $(p_1, p_2, \ldots, p_{n-1})$ in the following way. Since T is a tree with at least two vertices, we know that T has a leaf by Proposition 6.3.6. Let a_1 be the smallest label of a leaf, and let p_1 be its unique neighbor. Now if we delete a_1 from T, then we know from Proposition 6.3.7 that $T - a_1$ is also a tree. If this tree has at least two vertices, then let a_2 be the smallest label of a leaf, and let p_2 be its unique neighbor. Continue until we end with a unique vertex. Once we have completed this process, list the sequences on top of each other as follows:

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_{n-2} & p_{n-1} \end{array}$$

Notice that the $n-1$ edges of T will be:

$$\{a_1, p_1\}, \{a_2, p_2\}, \ldots, \{a_{n-1}, p_{n-1}\}.$$

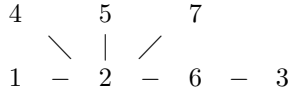
For example, given our tree



we obtain the following two sequences:

$$\begin{array}{cccccc} 2 & 4 & 5 & 6 & 3 & 1 \\ 4 & 3 & 1 & 3 & 1 & 7 \end{array}$$

For another example, given the tree



we obtain the following two sequences:

1	3	4	5	6	2
2	6	2	2	2	7

We have the following properties:

Proposition 6.3.16. *Let T be a tree with vertex set $[n]$.*

1. *$a_1, a_2, \ldots, a_{n-1}$ is a permutation of $[n-1]$.*
2. *$p_{n-1} = n$.*
3. *For all $k \in [n]$, $d(k)$ is the number of times that k occurs in $a_1, a_2, \ldots, a_{n-1}, p_1, p_2, \ldots, p_{n-1}$.*
4. *For all $k \in [n]$, $d(k)$ equals one plus the number of times that k occurs in $p_1, p_2, \ldots, p_{n-2}$.*

Proof. We have the following:

1. Certainly every number appears at most once as an a_i because it gets deleted after appearing there. Furthermore, at each stage we have a tree on at least 2 vertices, so it has at least 2 leaves by Corollary 6.3.9, and thus we never pick n.
2. Since there is no i with $a_i = n$, it follows that after $n-2$ stages we have two vertices, one of which is n. We pick the other as a_{n-1}, and thus $p_{n-1} = n$.
3. Since the edge set of T is
$$\{\{a_1, p_1\}, \{a_2, p_2\}, \ldots, \{a_{n-1}, p_{n-1}\}\},$$
we see that $d(k)$ equals the number of times that k occurs in the two lists.
4. This follows immediately from part (3) because every $k \in [n]$ occurs exactly once in $a_1, a_2, \ldots, a_{n-1}, p_{n-1}$ by parts (1) and (2).

□

There is some unnecessary information in the two sequences. As we've seen, we don't need p_{n-1} because we know what it will be. In fact, we also don't need the a_i at all, because we can recover them from the sequence $(p_1, p_2, \ldots, p_{n-2})$.

Definition 6.3.17. *Given a tree T with vertex set $[n]$, we call the sequence $(p_1, p_2, \ldots, p_{n-2})$ the* Prüfer code *of T.*

To see that we can recover the a_i from the Prüfer code, let's first consider an example. Suppose that T is a tree with and we know the following values:

?	?	?	?	?	?
5	1	7	5	2	?

Since there are 6 entries in each row, we know that T is a tree with vertex set $[7]$. We know that $p_6 = 7$, so we can fill that in:

?	?	?	?	?	?
5	1	7	5	2	7

Next, a_1 will be the smallest leaf. Using Proposition 6.3.16, we know that the degree of any vertex equals one plus the number of times that k occurs in $5, 1, 7, 5, 2$. Thus, we are looking for the least number that does not occur in this list. It follows that $a_1 = 3$, and we have:

3	?	?	?	?	?
5	1	7	5	2	7

To carry this forward, consider the following table:

vert	0	1	2	3	4	5
1	2	2	1	x	x	x
2	2	2	2	2	2	1
3	1	x	x	x	x	x
4	1	1	x	x	x	x
5	3	2	2	2	1	x
6	1	1	1	1	x	x
7	2	2	2	1	1	1

In the column labeled 0, we have put the degree of each vertex by adding 1 to the number of times that it occurs in $5, 1, 7, 5, 2$. Since 3 is the smallest leaf of T (as discussed above), and we know that its unique neighbor is 5, then when we delete vertex 3 from the tree, we also decrease the degree of 5 by one. The column labeled 1 gives the degrees after this deletion. The smallest leaf remaining can now be seen to be 4, which gives the value of a_2. Since the unique neighbor of 4 in that tree is 1, we decrease the degree of 1 by one after deletion to form the next column. Continuing this process, we arrive at the following sequence:

$$\begin{array}{cccccc} 3 & 4 & 1 & 6 & 5 & 2 \\ 5 & 1 & 7 & 5 & 2 & 7 \end{array}$$

The above procedure illustrates the following result.

Proposition 6.3.18. *Let T be a tree with vertex set $[n]$. From $p_1, p_2, \ldots, p_{n-2}$, we can completely determine p_{n-1} along with $a_1, a_2, \ldots, a_{n-1}$. Thus, if T and S are two trees with vertex set $[n]$ that have the same Prüfer code $(p_1, p_2, \ldots, p_{n-2})$, then they have same values for $p_{n-1}, a_1, a_2, \ldots, a_{n-1}$ as well, so they have the same edge set and hence are the same tree. It follows that the function that takes a tree with vertex set $[n]$ and produces the Prüfer code $(p_1, p_2, \ldots, p_{n-2})$ of T is injective.*

Proof. Since T is a tree with vertex set $[n]$, know that $p_{n-1} = n$. Using Proposition 6.3.16, we can compute $d(k)$ for each $k \in [n]$. Since a_1 is the smallest leaf, we know that a_1 must the smallest element of $[n]$ that is not in the set $\{p_1, p_2, \ldots, p_{n-2}\}$. In constructing these sequences, after we write down a_1 and p_1, we delete the leaf a_1 together with the unique edge incident to it. In the resulting tree, we no longer have a_1 as a vertex, and the degree of p_1 will be reduced by 1. Thus, the leaves of the resulting tree $T - a_1$ are then the numbers other than a_1 which do not occur in the list $p_2, p_3, \ldots, p_{n-2}$. In other words, a_2 must be the smallest element of $[n]$ that is not in the set $\{a_1\} \cup \{p_2, p_3, \ldots, p_{n-2}\}$. Now we delete a_2 and hence reduce the degree of p_2 by 1, so a_3 must be the smallest element of $[n]$ that is not in the set $\{a_1, a_2\} \cup \{p_3, \ldots, p_{n-2}\}$. In general, we can recursively reconstruct the sequence $a_1, a_2, \ldots, a_{n-1}$ because a_i is the smallest element of $[n]$ not in the set

$$\{a_1, a_2, \ldots, a_{i-1}\} \cup \{p_i, p_{i+1}, \ldots, p_{n-2}\},$$

which must exist since there are at most $n-2$ many numbers in this set. Therefore, we can reconstruct p_{n-1} along $a_1, a_2, \ldots, a_{n-1}$. The remaining statements follow. □

We've shown that every sequence $(p_1, p_2, \ldots, p_{n-2})$ occurs as the Prüfer code of at most one tree with vertex set $[n]$. We now show that every such code arises. Given any sequence $(p_1, p_2, \ldots, p_{n-2})$, the idea is to define the number p_{n-1} and $a_1, a_2, \ldots, a_{n-1}$ as in the proof of the previous result, and check that the resulting graph is a tree that produces the given code.

Proposition 6.3.19. *Let $n \in \mathbb{N}^+$, and suppose that $(q_1, q_2, \ldots, q_{n-2})$ is a sequence of integers with $1 \leq q_i \leq n$ for all i. Let $q_{n-1} = n$ and define $b_1, b_2, \ldots, b_{n-1}$ recursively by letting b_i be the smallest element of $[n]$ not in the set*

$$\{b_1, b_2, \ldots, b_{i-1}\} \cup \{q_i, q_{i+1}, \ldots, q_{n-2}\}$$

(which must exist since there are at most $n-2$ many numbers). If we let T be the graph with vertex set $[n]$ and edge set

$$\{\{b_1, q_1\}, \{b_2, q_2\}, \ldots, \{b_{n-1}, q_{n-1}\}\},$$

then T is a tree with Prüfer code $(q_1, q_2, \ldots, q_{n-2})$.

Proof. Notice that if $i < j$, then $b_j \neq b_i$ by the recursive definition of the sequence $b_1, b_2, \ldots, b_{n-1}$ (because b_i will be an element of the set $\{b_1, b_2, \ldots, b_{j-1}\}$. Thus, there are no repeated elements in the sequence $b_1, b_2, \ldots, b_{n-1}$. Furthermore, since for each i, the set

$$\{b_1, b_2, \ldots, b_{i-1}\} \cup \{q_i, q_{i+1}, \ldots, q_{n-2}\}$$

has at most $n-2$ many elements, and we choose b_i to be the smallest element of $[n]$ not in this set, it follows that $b_i \in [n-1]$ for all i. Putting these facts together, we conclude that $b_1, b_2, \ldots, b_{n-1}$ is a permutation of $[n-1]$. Now $b_i \neq q_i$ for all i by the recursive definition. Also, if $i < j$, then $\{b_i, q_i\} \neq \{b_j, q_j\}$ because $b_i \neq b_j$ from above and $b_i \neq q_j$ by the recursive definition. Thus, we can let T be the graph with vertex set $[n]$ and edge set

$$\{\{b_1, q_1\}, \{b_2, q_2\}, \ldots, \{b_{n-1}, q_{n-1}\}\}.$$

Since $\{b_i, q_i\} \neq \{b_j, q_j\}$ whenever $i < j$, it follows that T has $n-1$ many edges. We now show that T is acyclic. To see this, think about adding the edges of T in reverse order, i.e. add

$$\{b_{n-1}, q_{n-1}\}, \{b_{n-2}, q_{n-2}\}, \ldots, \{b_2, q_2\}, \{b_1, q_1\}$$

one at a time. At each stage, we claim that the resulting graph is acyclic. This is certainly true for the graph with just the single edge $\{b_{n-1}, q_{n-1}\}$. Suppose that the we know that the graph with edge set

$$\{\{b_{n-1}, q_{n-1}\}, \{b_{n-2}, q_{n-2}\}, \ldots, \{b_{i+1}, q_{i+1}\}\}$$

is acyclic. Now $b_i \notin \{b_{i+1}, \ldots, b_{n-2}, b_{n-1}\}$ from above, and $b_i \notin \{q_{i+1}, \ldots, q_{n-2}, q_{n-1}\}$ by definition. Thus, in the graph with edge set

$$\{\{b_{n-1}, q_{n-1}\}, \{b_{n-2}, q_{n-2}\}, \ldots, \{b_{i+1}, q_{i+1}\}, \{b_i, q_i\}\},$$

we see that b_i has degree 1. Now any cycle in this graph would have to include the new edge, but such a cycle would have to include the vertex b_i, which is impossible because the degree of b_i is 1 in this graph. Therefore, the graph with the new edge is acyclic. By induction, it follows that T is acyclic. Since T is an acyclic graph with n vertices and $n-1$ edges, Theorem 6.3.14 implies that T is a tree.

Now that we know that T is a tree, we just need to check that the process of constructing the sequences to obtain the Prüfer code (i.e. finding the smallest leaf, deleting it, etc.) to produce

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_{n-2} & p_{n-1} \end{array}$$

results in our sequences

$$\begin{array}{cccccc} b_1 & b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} \\ q_1 & q_2 & q_3 & \cdots & q_{n-2} & q_{n-1} \end{array}$$

Since the edge set of T equal $\{\{b_i, q_i\} : 1 \leq i \leq n-1\}$, we know that the degree of a vertex is just the number of times that it occurs in $b_1, b_2, \ldots, b_{n-1}, q_1, q_2, \ldots, q_{n-1}$. Furthermore, since we know that $q_{n-1} = n$ and that $b_1, b_2, \ldots, b_{n-1}$ is a permutation of $[n-1]$, it follows that the degree of any vertex is 1 plus the number of times that it occurs in $q_1, q_2, \ldots, q_{n-2}$. Now by definition of our recursive sequence, we have that b_1 is the smallest element of $[n]$ not in the set $\{q_1, q_2, \ldots, q_{n-2}\}$, which means that b_1 is the smallest vertex of degree

1, and hence the smallest leaf. It follows that $a_1 = b_1$ and hence $p_1 = q_1$ (because $\{b_1, q_1\}$ is an edge of T). Since b_1 is a leaf of T, we have that $T - b_1$ is a tree, and we know that it has edge set

$$\{\{b_2, q_2\}, \{b_3, q_3\}, \ldots, \{b_{n-1}, q_{n-1}\}\}.$$

In the resulting tree, we no longer have b_1 as a vertex, and the degree of q_1 will be reduced by 1. Thus, the leaves of the resulting tree $T - b_1$ are then the numbers other than b_1 which do not occur in the list $q_2, q_3, \ldots, q_{n-2}$. By definition of b_2, we conclude then that b_2 will be the smallest leaf in $T - b_1$. Thus, $a_2 = b_2$ and $p_2 = q_2$. Now delete b_2 and hence reduce the degree of q_2 by 1, and a similar argument shows that b_3 will be the smallest leaf in the resulting tree. In this way (or by induction), it follows that $a_i = b_i$ and $p_i = q_i$ for all i. □

We can now prove Cayley's Formula.

Proof of Theorem 6.3.15. Let $n \in \mathbb{N}^+$. If $n = 1$, then there is trivially only 1 tree with vertex set $[1]$, and we have $1^{1-2} = 1$. Suppose then that $n \geq 2$. Define a function from trees with vertex set $[n]$ to $\{1, 2, \ldots, n\}^{n-2}$ (i.e. the set of sequences of integers between 1 and n of length $n-2$) by assigning to each tree its Prüfer code. Notice that this function is injective by Proposition 6.3.18 and is surjective by Proposition 6.3.19. Therefore, the function is a bijection. Since $|\{1, 2, \ldots, n\}^{n-2}| = n^{n-2}$, there are n^{n-2} many trees with vertex set $[n]$. □

6.4 Minimum Weight Spanning Trees and Kruskal's Algorithm

Let $\mathbb{R}^{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$. Suppose that G is a graph, and $w \colon E \to \mathbb{R}^{\geq 0}$ is a function. Given $e \in E$, we call $w(e)$ the *weight* of the edge e, and we think about it as the "cost" of including edge e in our graph G. Based on these costs, we can choose to either include an edge or not. A natural question is how to include enough edges so that that we still have a connected graph, but so that we minimize the resulting cost. Since we will only think about deleting edges, we consider subgraphs H of G with $V_H = V_G$. Now given such a subgraph $H = (V_G, E_H)$ of G, we define

$$w(H) = \sum_{e \in E_H} w(e)$$

to be the sum of the weights of the edges that are in H. Notice that if our connected subgraph H includes a cycle, then we know that we can remove an edge of that cycle and still have a connected subgraph by Proposition 6.2.16. Since the weight of every edge is nonnegative, deleting such an edge does not increase the cost. In other words, we want what is called a *minimum weight spanning tree* of G. This leads to the following problem.

Question 6.4.1. *Given a connected graph G and a function $w \colon E \to \mathbb{R}^{\geq 0}$, how do we build a spanning tree T of G such that $w(T)$ is as small as possible?*

There are (at least) two natural attempts to build such a spanning tree by making a series of choices that seem reasonable:

1. *Idea 1:* Start with no edges, and include edges from G one at a time. We then have to ask ourselves which edge to include next. Since we start with no edges, the idea is to include one edge at a time without ever introducing a cycle. Since we are tying to minimize cost, we should pick the cheapest edge that does not introduce a cycle at each stage.

2. *Idea 2:* Start with all of the edges of G, and delete edges one at a time. We then have to ask ourselves which edge to delete next. We should only delete edges in cycles because we want a connected graph at the end. Since we are trying to minimize cost, we should pick the most expensive edge that is included in a cycle at each stage.

Notice that each of these are *greedy* algorithms. In other words, at each step we are picking a choice that looks best *locally* at that moment, without any assurance that it will produce a *globally* optimal solution in the end. In general, greedy algorithms do *not* produce globally optimal solutions. For example, if you want to climb a mountain, and you do it by always taking the one step that will increase your elevation most, then you may end up at the top of a tiny hill close by instead (because climbing the mountain may involve going down at some points). For another example, looking only one step ahead in chess may result in a move that looks excellent (say you kill a queen with your pawn), but it may be a globally bad move in that your opponent can checkmate you in the next move. We will see more precise examples of the failure of greedy algorithms in later sections. However, for the minimum weight spanning tree problem, it turns out that both of the above procedures do indeed produce minimum weight spanning trees!

We will study Idea 1, which as known as *Kruskal's algorithm*, because it is faster in practice (as we discuss below). However, let's first formalize this procedure more carefully. Suppose that we have a connected graph G and a function $w\colon E \to \mathbb{R}^{\geq 0}$. We build a sequence $H_0, H_1, H_2, \ldots, H_{n-1}$ of subgraphs of G with the following properties:

- $V_{H_i} = V_G$ for each i.
- H_i has i edges for each i.
- H_i is acyclic for each i.

We start by letting H_0 be the subgraph of G consisting of all of the vertices of G, but no edges. Suppose that we have constructed H_i with the above properties. Let

$$S_i = \{e \in E_G : e \notin E_{H_i} \text{ and } H_i + e \text{ is acyclic}\}.$$

We pick an element of S_i such that $w(e)$ is as small as possible (if there are multiple such edges in S_i, we pick an arbitrary one), and we let $H_{i+1} = H_i + e$. Once we've proceeding through each of the stages, we then take H_{n-1} as our answer. Although this ends the description of Kruskal's algorithm, there are several extremely important questions:

1. Why does Kruskal's algorithm never get stuck? In other words, why is each S_i nonempty?
2. Why is H_{n-1} a spanning tree of G?
3. Assuming that H_{n-1} is a spanning tree of G, why is it a minimum weight spanning tree? In other words, why is it the case that $w(H_{n-1}) \leq w(T)$ for all spanning trees T of G?
4. How do we implement it efficiently? After all, determining the elements of S_i by going through each edge e and checking if there is a cycle in $H_i + e$ seems to be costly.

In order to prove (1), and eventually to given efficient methods to answer (4), we use the following result.

Proposition 6.4.2. *Let G be a graph. Let $u, w \in V$ be distinct vertices that are not adjacent. Let e be a new edge with endpoints u and w.*

1. *If u and w are in the same connected component of G, then e is an element of a cycle of $G + e$.*
2. *If G is acyclic and u and w are in distinct connected components of G, then $G + e$ is acyclic.*

Proof. 1. Since u and w are in the same connected component of G, we can fix a u, w-path

$$u, f_1, v_1, f_2, v_2, \ldots, f_k, w$$

in G. Since this is a path, it is also a trail, and hence there are no repeated vertices or edges. We then have that

$$u, f_1, v_1, f_2, v_2, \ldots, f_k, w, e, u$$

is a closed walk with no repeated vertices or edges (because $e \notin E_G$ and hence $e \neq f_i$ for all i), so Proposition 6.2.12 implies that e is an element of a cycle of $G + e$.

2. We prove the contrapositive. Let G be an acyclic graph, and suppose instead that $G + e$ has a cycle C. Since G is acyclic, the cycle C must include the edge e. Using Proposition 6.2.12, there exists a closed walk without repeated vertices and edges corresponding to this cycle. By shifting the walk appropriately, we can write this closed walk as

$$w, e, u, f_1, v_1, f_2, v_2, \ldots, f_k, w$$

Notice that $f_i \neq e$ for all i because the walk has no repeated edges. It follows that

$$u, f_1, v_1, f_2, v_2, \ldots, f_k, w$$

is a u, w-walk in G, so u is in the same connected component of G.

□

Corollary 6.4.3. *Suppose that in the above algorithm we are at a stage i with $0 \leq i \leq n-2$ where we have that H_i is acyclic, has i edges, and satisfies $V_{H_i} = V_G$. We then have the following:*

1. *$S_i = \{e \in E_G : e \notin E_{H_i}$ and the endpoints of e are in distinct connected components of $H_i\}$.*
2. *$S_i \neq \emptyset$.*
3. *H_{i+1} is an acyclic graph with $i+1$ edges.*

Proof. 1. This follows immediately from the previous proposition and the fact that H_i is acyclic.

2. Notice that $|V_{H_i}| = |V_G| = n$, and $|E_{H_i}| = i \leq n-2$, so H_i is not connected by Corollary 6.3.13. Fix vertices u and w that are in distinct connected components in H_i. Since G is connected, there is a u, w-path in G, say

$$u = v_0, f_1, v_1, f_2, v_2, \ldots, f_k, v_k = w.$$

Since w and u are not in the same connected component in H_i, there is a smallest $i \geq 1$ such that v_i and u are not in the same connected component in H_i. We then have that v_{i-1} and u are in the same connected component of H_i. Thus, v_{i-1} and v_i are in distinct connected components in H_i, and so $f_i \in S_i$. Therefore, $S_i \neq \emptyset$.

3. By construction, there exists an edge $e \in S_i$ with $H_{i+1} = H_i + e$. We know that H_{i+1} is acyclic by definition of S_i. Since H_i has i edge, it follows that H_{i+1} has $i+1$ edges.

□

We've now seen that $S_i \neq \emptyset$ for each i, so the above algorithm never gets stuck and results in an acyclic subgraph H_{n-1} of G with $n-1$ edges, and such that $V_{H_{n-1}} = V_G$. Since H_{n-1} is an acyclic graph with $n-1$ edges, Theorem 6.3.14 implies that H_{n-1} is a tree. Therefore, H_{n-1} is a spanning tree of G. We now answer the third question about why it is a minimum weight spanning tree.

Theorem 6.4.4. *H_{n-1} is a minimum weight spanning tree of G.*

Proof. We first prove by induction on i that H_i is contained in some minimum weight spanning tree of G (i.e. we argue that at each stage there is still the possibility of extending to a spanning tree of minimum weight). For the base case of $i = 0$, we have that H_0 has no edges, so certainly H_0 is contained in a minimum weight spanning tree.

For the inductive step, suppose that $0 \leq i \leq n-2$ and the statement is true for i, i.e. that H_i is contained in some minimum weight spanning tree of G. Fix a minimum weight spanning tree T of G such that H_i is contained in T. Suppose that the algorithm picks edge e, so $e \in S_i$ and $w(e) \leq w(f)$ for all $f \in S_i$. We need to argue that $H_{i+1} = H_i + e$ is contained in some minimum weight spanning tree of G. We have two cases:

- If e is an edge of T, then $H_i + e$ is contained in T, which is a minimum weight spanning tree of G.
- Suppose that e is not an edge of T. By Theorem 6.3.14, the graph $T + e$ has a cycle. Fix such a cycle C of $T + e$, and notice that e must be an edge of C because T is acyclic. Now $H_i + e$ is also acyclic because $e \in S_i$, so C must contain an edge f that is not an edge of $H_i + e$. Since $f \neq e$, it follows that f must be an edge of T. Since f is an element of a cycle of the connected graph $T + e$, we may use Proposition 6.2.16 to conclude that $T + e - f$ is connected. Combining this with the fact that $T + e - f$ has the same number of edges as T, which is $n - 1$, we conclude that $T + e - f$ is a spanning tree of G by Theorem 6.3.14. We also have that $H_{n-1} + f$ is a subgraph of T, so $H_i + f$ is acyclic, and hence $f \in S_i$ as well. Therefore, we must have

  $$w(e) \leq w(f).$$

  Thus $w(e) - w(f) \leq 0$, and hence

  $$w(T + e - f) = w(T) + w(e) - w(f) \leq w(T).$$

  Now T is a minimum weight spanning tree, so we must have $w(T + e - f) = w(T)$, and hence that $T + e - f$ is also a minimum weight spanning tree. Since $H_{i+1} = H_i + e$ is contained in $T + e - f$, this completes the inductive step.

Since H_i is contained in a minimum weight spanning tree of G for all i with $0 \leq i \leq n - 1$, it follows that H_{n-1} is contained in some minimum weight spanning tree T of G. We already know from above that H_{n-1} is a spanning tree of G. Thus, by Theorem 6.3.14, $H_i + e$ is not a tree for any new edge e. It follows that $H_{n-1} = T$, and hence H_{n-1} is a minimum weight spanning tree of G. □

We've finally answered our first three questions about Kruskal's Algorithm, so we know that it does indeed produce a minimum weight spanning tree of G. How do we implement it efficiently? As mentioned above, the difficult part is computing the sets S_i. We know from Corollary 6.4.3 that

$$S_i = \{e \in E_G : e \notin E_{H_i} \text{ and the endpoints of } e \text{ are in distinct connected components of } H_i\},$$

so rather than checking whether an edge introduces a cycle, we can instead check if the endpoints are in distinct connected components of H_i. Of course, it is not immediately obvious how to do that. The essential idea is that we can keep track of the vertices in each of the connected components of the H_i throughout the algorithm by using the next result.

Proposition 6.4.5. *Let G be a graph, and let $u, w \in V$ be two vertices of G such that u and w are in distinct connected components of G. Consider the graph $G' = G + e$ where e is a new edge with endpoints u and w. Let $x \sim_G y$ mean that there is an x, y-walk in G and let $x \sim_{G'} y$ mean that there is an x, y-walk in G'. Moreover, for each $v \in V$, let C_v be the equivalence class of v under $\sim_G$, and let C'_v be the equivalence class of v under $\sim_{G'}$. Thus, C_v consists of the vertices in the connected component of v in G, while C'_v consists of the vertices in the connected component of v in G'. We have the following:*

1. *If $y \notin C_u$ and $y \notin C_w$, then $C'_y = C_y$.*
2. *If either $y \in C_u$ or $y \in C_w$, then $C'_y = C_u \cup C_w$.*

Proof. First notice that for any $a, b \in V$, if $a \sim_G b$, then $a \sim_{G'} b$ because an a, b-walk in G is an a, b-walk in G'.

1. Suppose that $y \notin C_u$ and $y \notin C_w$, so $y \not\sim_G u$ and $y \not\sim_G w$.

   - We first show that $C_y \subseteq C'_y$. Let $z \in C_y$ be arbitrary. We then have that $y \sim_G z$, so $y \sim_{G'} z$ from above, and hence $z \in C'_y$.
   - We now show that $C'_y \subseteq C_y$. Let $z \in C'_y$ be arbitrary. We then have that $y \sim_{G'} z$ and so we can fix a y, z-path P in G'. We claim that P does not contain the edge e. Suppose instead that P does include the edge e. Notice that since P is a path, it is a trail, so it must include e only once. Thus, either u or w occurs just before e on the path P, and if we cut off the path at this point, then we would have either an y, u-path in G or a y, w-path in G. Thus, either $y \sim_G u$ or $y \sim_G w$, each of which contradict our assumption. Thus, e does not occur in P. It follows that P is an y, z-path in G, so $y \sim_G z$ and hence $z \in C_y$.

   Combining these, we conclude that $C'_y = C_y$.

2. Suppose that $y \in C_u$, so $u \sim_G y$.

   - We first show that $C_u \cup C_w \subseteq C'_y$. Let $z \in C_u \cup C_w$ be arbitrary.
     - *Case 1:* Suppose that $z \in C_u$. We then have $u \sim_G z$. Since $u \sim_G y$, we can use symmetry and transitivity of $\sim_G$ to conclude that $y \sim_G z$. Therefore, $y \sim_{G'} z$ from above, and hence $z \in C'_y$.
     - *Case 2:* Suppose that $z \in C_w$. We then have $w \sim_G z$, and so $w \sim_{G'} z$ from above. Since $u \sim_G y$, we also have $u \sim_{G'} y$ from above. Finally, notice that $u \sim_{G'} w$ via the new edge e. Using symmetry of $\sim_{G'}$, we have

       $$y \sim_{G'} u \sim_{G'} w \sim_{G'} z$$

       By transitivity of $\sim_{G'}$, if follows that $y \sim_{G'} z$, so $z \in C'_y$.

     Therefore, in either case we have $z \in C'_y$. It follows that $C_u \cup C_w \subseteq C'_y$.
   - We now show that $C'_y \subseteq C_u \cup C_w$. Let $z \in C'_y$ be arbitrary. We then have that $y \sim_{G'} z$, so we can fix a y, z-path P in G'.
     - *Case 1:* Suppose that P does not include the edge e. We then have that P is a y, z-path in G, so $y \sim_G z$. Since we also know that $u \sim_G y$, we can use transitivity of $\sim_G$ to conclude that $u \sim_G z$. In follows that $z \in C_u$, and hence $z \in C_u \cup C_w$.
     - *Case 2:* Suppose that P does include the edge e. Since P is path, we know that it is trail, and hence e occurs exactly once. On this path, we then have that u and w occur immediately before and after e. If u occurs immediately after e on P, then the portion of the path that starts with u and goes to the end is a u, z-path in G, so $u \sim_G z$ and hence $z \in C_u$. Similarly, if w occurs immediately after e on P, then we obtain a w, z-path in G, so $w \sim_G z$ and hence $z \in C_w$. Thus, we have $z \in C_u \cup C_w$.

     Therefore, in either case, we have $z \in C_u \cup C_w$. It follows that $C'_y \subseteq C_u \cup C_w$.

3. The case where $y \in C_w$, then the argument that $C'_y = C_u \cup C_w$ is completely analogous to the argument in (2).

□

We can use this result to efficiently implement Kruskal's Algorithm as follows. Suppose that G is a finite connected graph and $w\colon E \to \mathbb{R}^{\geq 0}$ is a weight function. First, sort the edges by weight in increasing order, and label all of the vertices with distinct numbers. Now go through the sorted list of edges in order once, and do the following for each edge. Suppose that we are examining edge e with endpoints u and w. If u and w have the same label, do nothing but move on to the next edge. Suppose that u and w have distinct labels. We then have that u and w are in two distinct connected components of the current H_i, so $e \in S_i$, and it is straightforward to check that it will be an element of S_i of minimum weight (because the edges are sorted and we're going through them in order). Thus, we add e to H_i to form H_{i+1}, and update the labels so that we "merge" all vertices that have a label equal to one of the labels of u and w (i.e. change all of the labels of vertices that have the same label as w to now all have the same label as u).

6.5 Vertex Colorings and Bipartite Graphs

Definition 6.5.1. *Let G be a graph and let $k \in \mathbb{N}^+$.*

- *A function $c\colon V \to [k]$ is a* k-coloring *of (the vertices of) G.*
- *We say that a coloring $c\colon V \to [k]$ is* proper *if $c(u) \neq c(w)$ whenever $u, w \in V$ are distinct adjacent vertices.*
- *We define $\chi(G)$, the* chromatic number *of G, to be the smallest $k \in \mathbb{N}^+$ such that G has a proper k-coloring.*

If G is a graph and $k \in \mathbb{N}^+$, then saying that $\chi(G) \leq k$ is equivalent to saying that there is a proper k-coloring of G. Notice also that if $k \leq \ell$, then any proper k-coloring of a graph G is automatically a proper ℓ-coloring of G. Thus, to prove that $\chi(G) = k$, we need to do two things:

- Show that there does indeed exist a proper k-coloring of G (this shows that $\chi(G) \leq k$).
- Show that there does *not* exist a proper $(k-1)$-coloring G (this shows that $\chi(G) \not\leq k-1$).

Notice that $\chi(G) = 1$ if and only if G has no edges. For a more interesting example, the chromatic number of a cycle graph on 5 vertices is 3, i.e. $\chi(C_5) = 3$. To see that $\chi(C_5) \leq 3$, consider the following coloring $c\colon [5] \to [3]$ is a proper coloring of C_5:

- $c(1) = 1$
- $c(2) = 2$
- $c(3) = 1$
- $c(4) = 2$
- $c(5) = 3$

To show that $\chi(C_5) \not\leq 2$, we need to show that there is no proper 2-coloring of C_5. Consider a supposed proper coloring $c\colon [5] \to [2]$ of C_5. We must have $c(1) \neq c(2)$ and $c(2) \neq c(3)$, so since the codomain of c has 2 elements, we must have $c(1) = c(3)$. Similarly, we have $c(3) \neq c(4)$ and $c(4) \neq c(5)$, so $c(3) = c(5)$. Therefore, we would need to have that $c(1) = c(5)$, contradicting the fact that 1 and 5 are adjacent. Thus, $\chi(C_5) \not\leq 2$, and hence $\chi(C_5) = 3$.

How can we attempt to give a proper coloring of a finite graph G? One idea is do a *greedy* coloring. That is, fix an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G, say $v_1, v_2, \ldots, v_n$. Now color the vertices in order by giving vertex v_i the least color that is possible. By formalizing this, we obtain the following result.

Proposition 6.5.2. *$\chi(G) \leq \Delta(G) + 1$ for all finite graphs G, where $\Delta(G)$ is the largest degree of a vertex of G.*

Proof. Let G be a finite graph. Fix an ordering $v_1, v_2, \ldots, v_n$ of the vertices. We now define a coloring $c\colon V \to \mathbb{N}^+$ of the vertices in order recursively as follows. Let $c(v_1) = 1$. At stage $k+1$, once we've colored $v_1, v_2, \ldots, v_k$, let

$$S_{k+1} = \{i \in [k] : v_i \text{ and } v_{k+1} \text{ are adjacent}\},$$

and notice that $|S_{k+1}| \leq \Delta(G)$. Define $c(v_{k+1})$ to be the least element of

$$\mathbb{N}^+ \backslash \{c(v_i) : i \in S_{k+1}\}.$$

By doing this, we obtain a proper coloring c of G such that $c(v_i) \leq \Delta(G) + 1$ for all i because at each stage, the set $\{c(v_i) : i \in S_{k+1}\}$ that we've removed has at most $\Delta(G)$ many elements. Since c is a proper coloring of G using at most $\Delta(G) + 1$ many colors, we conclude that $\chi(G) \leq \Delta(G) + 1$. □

Notice that it is certainly possible that $\chi(G) < \Delta(G) + 1$. For example, if G is a graph with vertex set $[n]$, where n is adjacent to all other vertices but there are no other edges, then $\Delta(G) = n - 1$ but $\chi(G) = 2$. Thus, the above upper bound can be a *very* bad upper bound in some cases. One may object that the greedy coloring described in the above proof does not necessarily use $\Delta(G) + 1$ many colors. For example, in our example G, a greedy coloring using any ordering of the vertices only actually uses 2 colors. This is true, but there is an even deeper problem. Although the greedy coloring does indeed give a proper coloring of G (and leads to above inequality), using the greedy coloring on a particular ordering of the vertices may *not* produce a coloring using exactly $\chi(G)$ many colors. For example, consider the graph P_4 having vertex set $[4]$ and edge set $\{\{1,2\},\{2,3\},\{3,4\}\}$. If one uses the ordering $1, 2, 3, 4$ of the vertices, then indeed the greedy coloring uses 2 colors. However, if one uses the ordering $1, 4, 2, 3$ of the vertices, then the greedy coloring uses 3 colors.

Definition 6.5.3. *A graph G is* bipartite *if it has a proper 2-coloring, i.e. if $\chi(G) \leq 2$. Equivalently, G is bipartite exactly when it is possible to partition V into two disjoint sets A and B such that every edge of G has one endpoint in A and one endpoint in B (so no edge of G has endpoints in the same set).*

Theorem 6.5.4. *Let G be a graph. The following are equivalent:*

1. *G is bipartite.*
2. *G has no cycles of odd length.*
3. *G has no closed walks of odd length.*

Proof.

- (1) ⇒ (2): Assume that G is bipartite, and fix a proper coloring $c\colon V \to [2]$ of G. Suppose, for the sake of obtaining a contradiction, that G has a cycle of odd length. Fix a closed walk

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

without repeated edges or vertices (other than $v_0 = v_k$) and such that k is odd. Note that $k \geq 3$ because there are no loops in G, so we can fix $m \in \mathbb{N}^+$ with $k = 2m + 1$. We then must have $c(v_0) \neq c(v_1)$ and $c(v_1) \neq c(v_2)$, so since the codomain of c has 2 elements, it follows that $c(v_0) = c(v_2)$. A similar argument shows that $c(v_2) = c(v_4)$, and hence we conclude that $c(v_0) = c(v_4)$. In general, a simple induction shows that $c(v_0) = c(v_{2\ell})$ whenever $0 \leq \ell \leq m$. In particular, since $2m = k - 1$, we conclude that $c(v_0) = c(v_{k-1})$. Since $v_k = v_0$, this implies that $c(v_k) = c(v_{k-1})$, contradicting the fact that c is proper coloring of G. Therefore, G has no cycles of odd length.

- (2) ⇒ (3): We prove the contrapositive. Suppose then G has a closed walk of odd length. Fix an odd length closed walk of smallest possible length, say it is

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

with $v_0 = v_k$. Notice that $k \geq 3$ because G has no loops. We claim that there are no repeated vertices.

  - Suppose that there exists i with $0 < i < k$ such that $v_0 = v_i$. We then have $v_i = v_k$ as well, so

$$v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i$$

and

$$v_i, e_{i+1}, v_{i+1}, \ldots, v_{k-1}, e_k, v_k$$

are both closed walks. Since the sum of the lengths of these walks is k, which is odd, either i is odd or $k - i$ is odd. In either case, we have produced a closed walk in G of shorter odd length, which is a contradiction.

  - Suppose that there exists i, j with $0 < i < j < k$ and such that $v_i = v_j$. We then have that

$$v_i, e_{i+1}, v_{i+1}, \ldots, v_{j-1}, e_j, v_j$$

and

$$v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, e_{j+1}, v_{j+1}, \ldots, v_{k-1}, e_k, v_k$$

are both closed walks. Since the sum of the lengths of these walks is k, which is odd, either $j - i$ is odd or $k - (j - i)$ is odd. In either case, we have produced a closed walk in G of shorter odd length, which is a contradiction.

Both cases lead to a contradiction, so our shortest closed walk of odd length has no repeated vertices and length at least 3. Therefore, G has a cycle of odd length by Proposition 6.2.13.

- (3) ⇒ (1): Suppose that G has no closed walks of odd length. We first show that each connected component of G is bipartite. Consider an arbitrary connected component H of G. Fix some $z \in V_H$, and define a coloring $c\colon V_H \to [2]$ as follows. Given $v \in V_H$, fix a shortest possible z, v-path in G, and define $c(v) = 1$ if this path has even length, and $c(v) = 2$ if this path has odd length. We claim that c is a proper coloring of H. To see this, suppose that $u, w \in V_H$ are adjacent and $c(u) = c(w)$. We then have that a shortest possible z, u-path in H and a shortest possible z, w-path in H both have the same parity (either both even or both odd), so the sum of their lengths is even. Thus, if we follow a shortest z, u-path, then take then edge $\{u, w\}$ and then follow a shortest z, w-path backwards, we obtain a closed walk in G of odd length, which is a contradiction. Thus, H is bipartite.

  Since each of the connected components of G is bipartite, we can fix proper two colorings $c_H : V_H \to [2]$ of each connected component H of G. If we define $c\colon V_G \to [2]$ by letting $c(v) = c_H(v)$ for the unique connected component H containing v, then c is a proper coloring of G because two vertices in distinct connected components are not adjacent. Therefore, G is bipartite.

□

We know that a graph with n vertices has at most $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$ many edges. How many edges can a bipartite graph with n vertices have? A natural way to build a complete bipartite graph on n vertices is to look at $K_{m,n-m}$ for $m \in \{0, 1, \ldots, n\}$. Each of these graphs is bipartite (by coloring the vertices $\{1, 2, \ldots, m\}$ one color and the vertices $\{m+1, m+2, \ldots, n\}$ the other), and has all possible edges between the two sets corresponding to the color classes, which is why we called $K_{m,n}$ a *complete bipartite graph*. A straightforward calculation shows that $K_{m,n-m}$ has $m(n-m)$ many edges.

More generally, suppose that G is a bipartite graph with n vertices. Suppose that we fix a proper 2-coloring of G. Suppose that one of the color classes has size m. Then the other color class must have size $n-m$, so the number of edges in G is at most $m(n-m)$. Thus, in order to determine the maximum number of edges that a bipartite graph with n vertices can have, we need only figure out that largest possible value of $m(n-m)$ as we let $m \in \mathbb{N}$ vary between $0 \leq m \leq n$.

Suppose that $n \in \mathbb{N}^+$. If n is even and we write $n = 2m$, then $K_{m,m}$ is a bipartite graph with $m^2 = (\frac{n}{2})^2 = \frac{n^2}{4}$ many edges. If n is odd and we write $n = 2m+1$, then $K_{m+1,m}$ is a bipartite graph with

$$m(m+1) = \frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4}$$

many edges. Thus, in either case, we get

$$\left\lfloor \frac{n^2}{4} \right\rfloor$$

many edges. We now argue that this is best possible.

Theorem 6.5.5. *Let $n \in \mathbb{N}^+$.*

- *If n is even, say $n = 2k$ with $k \in \mathbb{N}^+$, then the maximum number of edges that a bipartite graph with n vertices can have is*
$$\frac{n^2}{4} = k^2.$$
- *If n is odd, say $n = 2k+1$ with $m \in \mathbb{N}$, then the maximum number of edges that a bipartite graph with n vertices can have is*
$$\frac{n^2-1}{4} = k^2 + k.$$

Thus, in either case, the maximum number of edges that a bipartite graph with n vertices can have is $\lfloor \frac{n^2}{4} \rfloor$.

Proof. Let G be a bipartite graph with n vertices. Since G is bipartite, we can fix a proper 2-coloring of the vertices of G. Let A be the set of vertices given one color, and and let B be the set of vertices given the other color. Let $m = |A|$, so $|B| = n - m$. Each vertex in A is then adjacent to at most $|B|$ many vertices, so the number of edges in G is at most $|A| \cdot |B| = m(n-m)$. Thus, if we determine the maximum values of $m(n-m)$ as we let m vary in the set $\{0, 1, 2, \ldots, n\}$, then we will have a bound on the number of edges in any bipartite graph. In order to maximize this discrete function of m, we examine the continuous (indeed differentiable) function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x(n-x) = nx - x^2.$$

We want to maximize $f(x)$ on the closed interval $[0, n]$. We have

$$f'(x) = n - 2x,$$

so $f'(x) > 0$ on $[0, \frac{n}{2}]$ and $f'(x) < 0$ on $[\frac{n}{2}, n]$. Thus, $f(x)$ is increasing on $[0, \frac{n}{2}]$ and decreasing on on $[\frac{n}{2}, n]$. Now if n is even, then $\frac{n}{2} \in \mathbb{N}$ with $0 \leq \frac{n}{2} \leq n$, and so the maximum occurs at $\frac{n}{2}$ with

$$f\left(\frac{n}{2}\right) = \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$$

Suppose that n is odd. Although the maximum of $f(x)$ occurs at $\frac{n}{2}$, this is not a natural number. However, $\frac{n-1}{2}$ is that largest natural number in the closed interval $[0, \frac{n}{2}]$, so as $f(x)$ is increasing on $[0, \frac{n}{2}]$, we conclude

that the largest value of f at a natural number in the interval $[0, \frac{n}{2}]$ is

$$\begin{aligned} f\left(\frac{n-1}{2}\right) &= \frac{n-1}{2} \cdot \frac{n+1}{2} \\ &= \frac{n^2-1}{4}. \end{aligned}$$

Similarly, $\frac{n+1}{2}$ is that smallest natural number in the closed interval $[\frac{n}{2}, n]$, so as $f(x)$ is decreasing on $[\frac{n}{2}, n]$, we conclude that the largest value of f at a natural number in the interval $[\frac{n}{2}, n]$ is

$$\begin{aligned} f\left(\frac{n+1}{2}\right) &= \frac{n+1}{2} \cdot \frac{n-1}{2} \\ &= \frac{n^2-1}{4}. \end{aligned}$$

Thus, the maximum value of $m(n-m)$ across all m in the set $\{0, 1, 2, \ldots, n\}$ equals $\frac{n^2-1}{4}$. □

Since the previous theorem gives an upper bound on the number of edges in a bipartite graph, it also gives an upper bound on the number of edges in a graph that does not contain an odd cycle (so a 3-cycle, a 5-cycle, etc.). Somewhat surprisingly, if want to determine the maximum number of edges in a graph that does not contain a triangle (i.e. a 3-cycle), we obtain the exact same upper bound. To prove this, we use a different inductive approach.

Theorem 6.5.6. *We have the following.*

1. *If $k \geq 2$ and G is a graph with $n = 2k$ vertices and at least $k^2 + 1$ many edges, then G contains a triangle.*
2. *If $k \geq 1$ and G is a graph with $n = 2k+1$ vertices and at least $k^2 + k + 1$ many edges, then G contains a triangle.*

Proof. 1. We prove this by induction on k.

- *Base Case:* Suppose that $k = 2$. Let G be a graph with $4 = 2 \cdot 2$ vertices and at least $2^2 + 1 = 5$ edges. We know that G is not bipartite by the previous theorem, so it has an odd cycle. Such a cycle must have length 3 (because there are only 4 vertices), so G contains a triangle.
- *Inductive Step:* Suppose that the statement is true for a fixed $k \geq 2$. Let G be a graph with $2(k+1) = 2k+2$ many vertices and at least $(k+1)^2 + 1$ many edges. Fix an edge in G, and call its endpoints u and w. If u and w have a common neighbor in G, then we have a triangle and we are done. Suppose then that u and w do not have a common neighbor in G. For each of the other $2k$ vertices, at most one of u or w is adjacent to it, so there are at most $2k$ many other edges incident to at least one of u or w, not counting the edge $\{u, w\}$ itself. Thus, if we delete the two vertices u and w to form $G - \{u, w\}$, then the resulting graph has at most $2k+1$ many fewer edges than G. It follows that $G - \{u, w\}$ has at least

$$\begin{aligned} (k+1)^2 + 1 - (2k+1) &= k^2 + 2k + 1 + 1 - 2k - 1 \\ &= k^2 + 1 \end{aligned}$$

  many edges. By induction, $G - \{u, w\}$ has a triangle, so G does.

The result follows by induction.

2. We also prove this by induction on k.

- *Base Case:* Suppose that $k = 1$. Let G be a graph with $3 = 2 \cdot 1 + 1$ vertices and at least $1 \cdot 2 + 1 = 3$ edges. We then have that G is a triangle, so we are done.
- *Inductive Step:* Suppose that the statement is true for a fixed $k \geq 2$. Let G be a graph with $2(k+1)+1 = 2k+3$ many vertices and at least $(k+1)(k+2)+1$ many edges. Fix an edge in G, and call its endpoints u and w. If u and w have a common neighbor in G, then we have a triangle and we are done. Suppose then that u and w do not have a common height in G. For each of the other $2k+1$ vertices, at most one of u or w is adjacent to it, so there are at most $2k+1$ many other edges incident to one of u or w, not counting $\{u, w\}$ itself. Thus, if we delete the two vertices u and w to form $G - \{u, w\}$, then the resulting graph has at most $2k+2$ many fewer edges than G. It follows that $G - \{u, w\}$ has at least

$$\begin{aligned}(k+1)(k+2)+1-(2k+2) &= k^2+3k+2+1-2k-2\\ &= k^2+k+1\\ &= k(k+1)+1\end{aligned}$$

  many edges. By induction, $G - \{u, w\}$ has a triangle, so G does.

The result follows by induction.

□

6.6 Matchings

Definition 6.6.1. *A* matching *in a graph is a set of edges such that no two distinct edges have a common endpoint.*

We will often be interested in matchings in bipartite graphs. The idea is that one side represents people and the other jobs/tasks.

Definition 6.6.2. *Let G be a graph and let M be a matching in G.*

- *Let $S \subseteq V_G$. We say that M* saturates *S if every element of S is an endpoint of some edge in M.*
- *We say that M is a* perfect matching *if M saturates V, i.e. every vertex in G appears as an endpoint of some edge in M.*
- *We say that M is a* maximal matching *in G if $M \cup \{e\}$ is not a matching for every edge $e \in E \backslash M$.*
- *We say that M is a* maximum matching *if it has at least as many edges as any other matching, i.e. if $|N| \leq |M|$ for all matchings N of G.*

Notice that any maximum matching in a graph G is a maximal matching. However, the converse is not true. For example, consider the graph P_4, so the vertex set is $[4] = \{1, 2, 3, 4\}$ and the edge set is $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Notice that $M = \{\{2, 3\}\}$ is a maximal matching, but it is not a maximum matching because $M' = \{\{1, 2\}, \{3, 4\}\}$ is a matching with strictly more elements. Although it is relatively easy to determine if a given matching is a maximal matching (simply look through all of the edges in turn and check if each of their endpoints are saturated by M), it seems harder to determine if a matching is a maximum matching. The following definition will be essential to help us efficiently determine if a matching is a maximum matching.

Definition 6.6.3. *Let M be a matching in a graph G.*

- *An* M-alternating path *in* G *is a path*

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

in G *where the* e_i *alternate between elements of* M *and element of* $E \backslash M$ *(i.e. either* $e_1, e_3, e_5, \ldots$ *are all elements of* M *and* $e_2, e_4, e_6, \ldots$ *are all elements of* $E \backslash M$*, or* $e_1, e_3, e_5, \ldots$ *are all elements of* $E \backslash M$ *and* $e_2, e_4, e_6, \ldots$ *are all elements of* M*).*

- *An* M-augmenting path *in* G *is an* M*-alternating path*

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

in G *where* $k \geq 1$ *and both* v_0 *and* v_k *are* M*-unsaturated (i.e. not the endpoint of* any *edge in* M*).*

Notice that in an M-augmenting path, the first and last edges e_1 and e_k must both be elements of $E \backslash M$. However, an M-augmenting path is more than just an M-alternating path with this property because we require that *no* edge of M is incident to either v_0 or v_k, not just e_1 and e_k themselves. Before establishing our major theorem about M-augmenting paths, we first prove a useful lemma.

Lemma 6.6.4. *Let* G *be a finite graph with* $d(v) \leq 2$ *for all* $v \in G$*. We then have that every connected component of* G *is either a path or a cycle.*

Proof. Let H be an arbitrary connected component of G. Since G is finite, we know that H is finite, so we can fix a longest possible path

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

in H (note that possibly $k = 0$, in which case we have a trivial path). Since paths are trails, we know that the edges in this path are all distinct. For each i with $1 \leq i \leq n-1$, we have that v_i is incident to the two distinct edges e_i and e_{i+1}. Since $d(v_i) \leq 2$ for each i with $1 \leq i \leq k-1$, it follows that v_i is incident to no edges of G, other than e_i or e_{i+1}. We now have a few cases:

- *Case 1:* Suppose that v_0 and v_k are incident to no edges besides e_1 and e_k, respectively. We then have H consists of the vertices and edges on this path, and no other vertices/edges, so H is a path.

- *Case 2:* Suppose then v_0 is incident to another edge f. Notice that the other endpoint of f must be some v_i, because otherwise we could extend our path to a longer one in H. Now this other endpoint cannot be a v_i with $1 \leq i \leq k-1$ from above, so the other endpoint must be v_k. We then have that

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k, f, v_0$$

is a closed walk without repeated vertices/edges. Furthermore, no other edge can be incident to either v_0 or v_k because $d(v_0) \leq 2$ and $d(v_k) \leq 2$. We now have a cycle in H, and furthermore no other vertices or edges can be in H. Therefore, H is a cycle.

- *Case 3:* Suppose then v_k is incident to another edge f. Then following the argument in Case 2, the other endpoint of f must be v_0, and we conclude that H is a cycle.

□

We are now ready to prove our fundamental theorem about augmenting paths.

Theorem 6.6.5. *Let* M *be a matching in a finite graph* G*. The following are equivalent:*

1. M *is a maximum matching.*
2. *There is no* M*-augmenting path in* G*.*

Proof. We prove the contrapositive of each direction, so we show that M is *not* a maximum matching if and only if there exists an M-augmenting path in G.

Suppose first that there does exist an M-augmenting path in G, say it is

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

with $k \geq 1$. Since v_0 and v_k are both M-unsaturated, we know that $e_1 \in E\backslash M$ and $e_k \notin E\backslash M$. Since this path is M-alternating (because it is M-augmenting), we must have that $e_i \in E\backslash M$ for all odd i, and $e_i \in M$ for all even i. In particular, it must be the case that k is odd. Let

$$M' = (M\backslash\{e_i : i \text{ is even}\}) \cup \{e_i : i \text{ is odd}\}$$

Notice that if $1 \leq i \leq k-1$, then v_i is incident to a unique edge in M' because it is incident to a unique edge of M. Also, since v_0 and v_k are both M-unsaturated, they are each incident to a unique edge in M'. Finally, for any vertex u not equal to any v_i, we have that u is incident to at most one edge in M' because it is incident to at most one edge in M. It follows that M' is matching in G. Now since k is odd, we have that $|M'| = |M| + 1$, so M' is matching in G with more elements than M. Therefore, M is not a maximum matching.

Suppose conversely that M is not a maximum matching. Fix a maximum matching M' in G, and notice that $|M| < |M'|$. Consider the subgraph H of G with vertex set $V_H = V_G$ and edge set the symmetric difference $E_H = M \Delta M'$, i.e. $E_H = (M\backslash M') \cup (M'\backslash M)$. Thus, an edge $e \in E_G$ appears in E_H when it is in exactly one of M or M'. Notice that since $|M'| > |M|$, we have

$$|M'\backslash M| > |M\backslash M'|.$$

For any $v \in V_H$, we know that v is incident to at most one edge in M and at most one edge in M', so we have that $d_H(v) \leq 2$ for all $v \in V_H$. Thus, by Lemma 6.6.4, each connected component of H is either a path or a cycle. Now any cycle in H must alternate edges between elements of $M\backslash M'$ and elements of $M'\backslash M$ because no vertex is incident two edges in M, and no vertex is incident to edges in M'. Hence, each connected component in H that is a cycle has even length and an equal number of edges in both $M\backslash M'$ and $M'\backslash M$. Therefore, since $|M'\backslash M| > |M\backslash M'|$, there must exist a connected component in H that is path with strictly more edges in $M'\backslash M$ than in $M\backslash M'$. Fix such a path

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

Since each vertex is incident to at most edge of M and at most one edge of M', it follows that the e_i alternate between edges of $M'\backslash M$ and edges of $M\backslash M'$. Furthermore, since this path has strictly more edges in $M'\backslash M$ than in $M\backslash M'$, we must have $e_1, e_k \in M'\backslash M$. Thus, our path is an M-alternating path with $e_1, e_k \notin M$. Moreover, both v_0 and v_k must be M-unsaturated, because if either was incident to an edge in M, that edge would be in $M\backslash M'$ (since it is not e_1 and not e_k), which implies that it would appear in H, contradicting the fact that our path is a component in H. Therefore, we have shown the existence of an M-augmenting path in G. □

In many cases of applied interest, we are looking for matchings in bipartite graphs. Suppose then that G is a bipartite graph. Fix a proper 2-coloring of G, and let X and Y be the corresponding color classes. We want to find a simple necessary and sufficient condition for when there is a matching G that saturates X.

Definition 6.6.6. *Let G be graph. Given a set $T \subseteq V$, we define*

$$N(T) = \{v \in V : v \text{ is adjacent to some element of } T\}.$$

In other words, $N(T)$ is the set of neighbors *of elements of T.*

Theorem 6.6.7 (Hall's Marriage Theorem)**.** *Let G be a finite bipartite graph. Fix a proper 2-coloring of G, and let X and Y be the corresponding color classes. There exists a matching of G that saturates X if and only if $|T| \leq |N(T)|$ for all $T \subseteq X$.*

Proof. Suppose first that there exists a matching of G that saturates X, and fix such a matching M. Let $T \subseteq X$ be arbitrary. Since M saturates X and $T \subseteq X$, we can define a function $f\colon T \to N(T)$ by letting $f(x)$ be the unique vertex that x is matched to in M. Since M is a matching, this function is injective, so $|T| \leq |N(T)|$.

Suppose conversely that there does *not* exist a matching of G that saturates X. We build a set $T \subseteq X$ with $|T| > |N(T)|$. First, fix a maximum matching M of G. Now M does not saturate X by assumption, so we can fix an $a \in X$ that is M-unsaturated. Consider the set of all M-alternating paths whose first vertex is a. Let B be the set of all endpoints of such paths that are elements of X, i.e.

$$B = \{x \in X : \text{There exists an } M\text{-alternating } a, x\text{-path in } G\}.$$

Similarly, let C be the set of all endpoints of such paths that are elements of Y, so

$$C = \{y \in Y : \text{There exists an } M\text{-alternating } a, y\text{-path in } G\}.$$

Thus, $B \subseteq X$, $C \subseteq Y$, and $a \in B$ (because the trivial path of just a is an M-alternating path). We have the following:

1. Every element of $B\backslash\{a\}$ is M-saturated and its matched partner is in C: Let $b \in B\backslash\{a\}$ be arbitrary. By definition of B, we can fix an M-alternating a, b-path P in G. Notice that P has even length of at least 2 because G is bipartite, $a, b \in X$, and $b \neq a$. Since the edges of P alternate between elements of $E\backslash M$ and M, and since P starts with an edge in $E\backslash M$ (because a is M-unsaturated), it follows that the last edge of P is an element of M. Thus, b is M-saturated. Furthermore, the penultimate vertex of P is matched to b, is an element of Y, and is the endpoint of the M-alternating path starting with a that is obtained by deleting the last vertex and edge of P. Therefore, the matched partner of b is an element of C.

2. Every element of C is M-saturated and its matched partner is in $B\backslash\{a\}$: Let $c \in C$ be arbitrary. By definition of C, we can fix an M-alternating a, c-path P in G. Notice that P has odd length because G is bipartite, $a \in X$, and $c \in Y$. Since the edges of P alternate between elements of $E\backslash M$ and M, and since P starts with an edge in $E\backslash M$ (because a is M-unsaturated), it follows that the last edge of this path is an element of $E\backslash M$. Furthermore, since M is a maximum matching, we know that P can not be an M-augmenting path by Theorem 6.6.5. Now P is M-alternating, and a is M-unsaturated, so it must be the case that c is M-saturated. Let $b \in X$ be the matched partner of c. Notice that b does not occur on P because c does not occur before the last vertex, and the last edge is an element of $E\backslash M$. Thus, if we add on the edge $\{c, b\}$ and the vertex b to the end of P, we obtain an M-alternating a, b-path in G, so $b \in B$. Also, notice that $b \neq a$ because a is M-unsaturated. Therefore, the matched parter of c is an element of $B\backslash\{a\}$.

3. $|C| = |B\backslash\{a\}|$: By (1), we can define a function from $B\backslash\{a\} \to C$ sending an element to its matched partner. This function is injective because M is a matching, so $|B\backslash\{a\}| \leq |C|$. Similarly, $|C| \leq |B\backslash\{a\}|$ by (2). It follows that $|C| = |B\backslash\{a\}|$.

4. $N(B) \subseteq C$: Let $y \in N(B)$ be arbitrary. Since $y \in N(B)$, we can fix $b \in B$ such that b is adjacent to y. Since $b \in B$, we can fix an M-alternating a, b-path P in G. If y is a vertex on P, then by cutting off P at y we obtain an M-alternating a, y-path in G, so $y \in C$. Suppose then that y is not a vertex on P. As in (1), notice that P has even length and the last edge of P is an element of M. Since y is not a vertex on P, and we know that the penultimate vertex of P is the matched partner of b, it follows that $\{b, y\} \notin M$. Thus, if we add on the edge $\{b, y\}$ and the vertex y to the end of P, we obtain an M-alternating a, y-path in G, so $y \in C$. It follows that $N(B) \subseteq C$.

Combining (3) and (4), we have

$$|N(B)| \leq |C| = |B \backslash \{a\}| = |B| - 1$$

Therefore, we may let $T = B$. □

Instead of thinking about finding a large matching, we now move on to consider finding a "good" matching in a certain sense. Suppose that we have two groups of n vertices and each one side has a ordering of the other in terms of preference. Thus, we have the complete bipartite graph $K_{n,n}$ where the vertex set is partitioned into two sets U (think of uppercase letters) and L (think of lowercase letters) of size n such that every vertex of U is adjacent to every vertex of L. Suppose furthermore that for each $A \in U$, we have a permutation of L, which we can think of as an ordering $<_A$ of L. Similarly, for each $x \in L$, we have a permutation of Y, which we can think of as an ordering $<_x$ of U. These orderings codify the preferences of the vertices. For example, if $U = \{A, B, C, D\}$ and $L = \{w, x, y, z\}$, we may have the following lists of preferences:

A	B	C	D
x	w	z	z
y	x	w	w
z	y	y	x
w	z	x	y

w	x	y	z
C	C	C	B
B	A	B	C
A	D	A	A
D	B	D	D

In this setting, there are of course many perfect matchings (in fact there are $n!$ many of them). However, we want to find a "good" matching. There are several ways one could define a notion of "good" (as many vertices as possible are paired with their first choice, as few vertices as possible are paired with their last choice, minimizing the sum of the ranks of the matched pairs, etc.), but we opt for a notion that tries to avoid "rogue" pairs.

Definition 6.6.8. *A matching is* stable *if there do not exist matched pairs $\{x, A\}, \{y, B\} \in M$ (where $A, B \in U$ and $x, y \in L$) such that $B <_y A$ and $x <_A y$, i.e. such that y prefers A to B and also A prefers y to x.*

If we think of our matching as providing marriages between uppercase people U and lowercase people L, then a stable matching is one where there does not exist two people who would prefer each other to their current spouses, thus warding off infidelity. Does every list of preferences have a stable matching? If so, is it unique? Also, how could we find one? There is a clever algorithm to form a matching that is useful in answering all of these questions. The idea is to pick one side and have it do a sequence of proposals to the other side. Say that we have the L vertices do the proposals to the U vertices. At the first stage, each L vertex approaches their first choice of a vertex U and proposes. Each U vertex that now has a proposal tells their favorite current suitor to return in the next round, and rejects the others, telling them to never return. We call the resulting pair *engaged*. In the next round, each engaged element of L returns to their engaged partner, and each element of L that is not currently engaged proposes to their highest choice of a vertex in U that has not yet rejected them. At this point, each U vertex (even if they are currently engaged) that now has a proposal tells their favorite current suitor/engaged partner to return in the next round, and also rejects the others (possibly including the current engaged partner) telling them to never return. As before, we call the resulting pairs engaged. We continue this process until we reach a round where everybody is engaged, and we take this matching. For example, consider the above lists of preferences:

A	B	C	D
x	w	z	z
y	x	w	w
z	y	y	x
w	z	x	y

w	x	y	z
C	C	C	B
B	A	B	C
A	D	A	A
D	B	D	D

With the elements of L proposing, we get the following run of the algorithm (where bold indicates the engaged element of L):

Round	A	B	C	D
1		**z**	**w**, x, y	
2	**x**	**y**, z	**w**	
3	**x**	**y**	w, **z**	
4	**x**	**w**, y	**z**	
5	**x**, y	**w**	**z**	
6	**x**	**w**	**z**	**y**

Thus, we get the following matching:

$$\{w, B\} \quad \{x, A\} \quad \{y, D\} \quad \{z, C\}.$$

We can also switch and have elements of U proposing, giving the following run of the algorithm:

Round	w	x	y	z
1	**B**	**A**		**C**, D
2	**B**, D	**A**		**C**
3	**B**	**A**, D		**C**
4	**B**	**A**	**D**	**C**

Notice that this results in the same matching. Of course, we now have a couple of questions. First, does this procedure always terminate in a matching? For example, is it possible that an element of the proposing set is rejected by everyone? If we know that the process does terminate in a matching, then the big question is whether the resulting matching stable. We start with the termination question. Suppose that the elements of L propose to the elements of U. By definition of the algorithm, we have the following properties:

1. If $x \in L$ is rejected by $A \in U$, then x never proposes to A again.
2. If $A \in U$ is engaged to an element of L at stage k, then A is engaged to some (possibly different) element of L at all later stages.

Furthermore, building on these facts, a closer look at the algorithm reveals that we also have the following properties:

3. If we fix $x \in L$, then the sequence of elements of U that x proposes to each day is decreasing (not necessarily strictly) through its preference list, and never skips over anybody on their list.
4. If we fix $A \in U$, then the sequence of elements of L that A is engaged to might start out empty, but then it is increasing (not necessarily strictly) though its preference list, and it might skip people on the list.

Now if we ever reach a stage where each element of U has a current proposal by some element of L, then we stop the algorithm and take the corresponding matching. With this in mind, we first argue that no element of L can be rejected by every element of U. To see this, suppose that we are at a stage where a given $x \in L$ gets rejected by the $(n-1)^{st}$ person on their list. By property 3 above, it follows that x has now been rejected by each of the first $n-1$ elements of their list. Now using property 2, it follows that on the next day, those $n-1$ people will each have a suitor, and then x will propose to the n^{th} person on their list. Thus, each of the n elements of U have a suitor, and since L also has n elements, it follows that each element of U has a unique suitor. Thus, the algorithm must terminate in a matching. Furthermore, since at least one rejection happens at each stage that does not produce the final matching, the above argument shows that algorithm terminates in at most $n(n-2)+2$ many steps (although this can be improved a bit). We now prove the algorithm produces a stable matching.

Theorem 6.6.9. *Suppose that the elements of L propose to the elements of U. The matching produced by the algorithm is stable.*

Proof. Suppose that the matching produces pairs $\{x, A\}$ and $\{y, B\}$. Suppose that y prefers A to B. Then at some stage, y must have proposed to A by property 3 above. Since y is not paired with A, it follows that A must have rejected y at some (possibly later) stage in favor of somebody that A preferred. Since the sequence of engagements of A only increases by property 4, it follows that A prefers x to y. □

A	B	C	D
x	w	z	z
y	x	w	w
z	z	y	x
w	y	x	y

w	x	y	z
C	D	C	B
B	A	B	C
A	C	A	A
D	B	D	D

With the lowercase letters proposing, we get the following run:

Round	A	B	C	D
1		**z**	**w**, y	**x**
2		y, **z**	**w**	**x**
3	**y**	**z**	**w**	**x**

Thus, we get the following matching:

$$\{w, C\} \quad \{x, D\} \quad \{y, A\} \quad \{z, B\}.$$

If we run it in the other order, we get

Round	w	x	y	z
1	**B**	**A**		**C**, D
2	**B**, D	**A**		**C**
3	**B**	A, **D**		**C**
4	**B**	**D**	**A**	**C**

Thus, we get the following matching:

$$\{w, B\} \quad \{x, D\} \quad \{y, A\} \quad \{z, C\}.$$

Both of these matchings are stable by the above argument. In particular, there can be more than one stable matching. This lead to the question of whether there is a "best" stable matching.

Definition 6.6.10. *Let $x \in L$ and let $A \in U$.*

- *We say that A is* feasible *for x if $\{x, A\}$ occurs in some stable matching. Similarly, we say that x is* feasible *for A if $\{x, A\}$ occurs in some stable matching.*
- *We say that A is* optimal *for x if A is the highest element of x's preference list that is feasible for x. Similarly, we say that x is* optimal *for A if x is the highest element of A's preference list that is feasible for A.*

Theorem 6.6.11. *If L does the proposing, then for all $x \in L$, the partner produced by the algorithm is optimal for x.*

Proof. We show that if x is rejected by A at some stage, then A is not feasible for x. We do this by induction on the stage of the construction:

- Suppose that A rejects x at the first stage. We need to show that A is not feasible for x. Let M be an arbitrary perfect matching containing $\{x, A\}$. Since A rejects x at the first stage, we know that A is engaged to some y at the end of the first stage, and so A prefers y to x. Fix B such that $\{y, B\} \in M$. We then have that A prefers y to x (from above) and y prefers A to B (since A was y's first choice as this is the first round), so M is not stable. Thus, any matching containing $\{x, A\}$ is not stable, so A is not feasible for x.

- Suppose now that we are at stage k, and we know that whenever $z \in L$ has been rejected by some $C \in U$ at a stage before k, then C is not feasible for z. Suppose now that A rejects x at stage k. We need to show that A is not feasible for x. Let M be an arbitrary perfect matching containing $\{x, A\}$. Since A rejects x at stage k, we know that A is engaged to some y at the end of stage k, and that A prefers y to x. Fix B such that $\{y, B\} \in M$. If B is not feasible for y, then M is not stable definition. Suppose then that B is feasible for y. Since y is engaged to A at stage k, we know that y was rejected by all elements of U above A in earlier rounds, and hence no element above A is feasible for y by induction. Since B is feasible for y, we must have that y prefers A to B. Combining this with the fact that A prefers y to x (from above), we conclude that M is not stable. Thus, any matching containing $\{x, A\}$ is not stable, so A is not feasible for x.

We have shown that if x is rejected by A at some stage, then A is not feasible for x. Since x is matched with the highest ranked element of U that does not reject x, it follows that the partner produced for x is optimal for x. □

Thus far, we've been working in the graph $K_{n,n}$ where each vertex on one side ranks the elements of the others. Suppose instead that we work in K_{2n} where each vertex ranks *all* other vertices. Think about this as taking a group of $2n$ people and trying to pair off roommates. Although this problem superficially seems completely analogous, the lack of two "sides" changes the situation dramatically. In fact, there may not be a stable matching! For example, suppose that $n = 2$, so we have 4 people who rank the other 3 as follows:

A	B	C	D
B	C	A	A
C	A	B	B
D	D	D	C

To see that there is no stable matching, simply look at who is matched with D (who is the lowest ranked vertex for all others).

- If we match $\{A, D\}$, then we must match $\{B, C\}$, and then A and C form an unstable pair.

- If we match $\{B, D\}$, then we must match $\{A, C\}$, and then A and B form an unstable pair.

- If we match $\{C, D\}$, then we must match $\{A, B\}$, and then B and C form an unstable pair.

Therefore, there is no stable matching.

6.7 Planar Graphs

Definition 6.7.1. *Let G be a graph. A* planar embedding *of G is way to draw G in the plane such that all vertices are represented by points, all edges are represented by continuous paths, and no two distinct edges cross (except at the endpoints). We say that G is* planar *if there exists a planar embedding of G.*

One can make this definition more formal be representing edges by continuous functions $g\colon [0, 1] \to \mathbb{R}^2$ such that $g(0)$ is one endpoint and $g(1)$ is the other endpoint. A careful treatment of this material relies

on some important properties of continuous functions, and hence relies on some analysis and topology. We (obviously) don't have those tools, so we will proceed a bit more intuitively. However, rest assured that all of these results can be made very precise with the appropriate tools.

Definition 6.7.2. *Given a planar embedding of a graph G, we divide the plane (minus the image of the embedding) into regions that we call* faces.

Theorem 6.7.3 (Euler's Formula)**.** *Let G be a planar embedding of a connected planar multigraph. If this embedding has n vertices, m edges, and c faces, then $n - m + c = 2$.*

Proof. The proof is by induction on m.

- *Base Case:* Suppose that $m = 0$. Since G is connected, we must have $n = 1$ and hence $c = 1$ as well. Notice that $2 - 1 + 1 = 2$.
- *Inductive Step:* Suppose the result is true for all connected planar multigraphs with m edges. Let G be a connected planar multigraph with n vertices, $m + 1$ many edges, and c faces. We have two cases:
  - *Case 1:* G has no cycles, so G is a tree. We then have $m = n - 1$ by Theorem 6.3.8 and $c = 1$ (because there are no cycles), so
  $$\begin{aligned} n - m + c &= n - (n - 1) + 1 \\ &= 2. \end{aligned}$$
  - *Case 2:* G has a cycle. Fix an edge e in a cycle, and let $G' = G - e$. Notice that G' is a connected planar graph (using Proposition 6.2.16) with n vertices and m edges. Furthermore, G' has $c - 1$ many faces because the faces on opposing sides of e become one face. By induction, we have
  $$n - m + (c - 1) = 2.$$
  Therefore
  $$n - (m + 1) + c = 2,$$
  and hence the statement is true for G.

  The result follows by induction.

□

Definition 6.7.4. *Suppose we have a planar embedding of a connected multigraph G. Given a face $f \in F$, we define the length of f, denoted $\ell(F)$ to be the length of the walk which traverses the boundary of the face. Notice that if F is on both "sides" of an edge, then that edge is counted twice.*

Proposition 6.7.5. *If G is a connected planar multigraph with m edges, then $\sum_{f \in F} \ell(f) = 2m$.*

Proof. Every edge has 2 "sides", so is counted twice on the left. □

Proposition 6.7.6. *Let G be a connected planar graph with n vertices and m edges. If $n \geq 3$, then $m \leq 3n - 6$.*

Proof. Suppose that $n \geq 3$. For each face $f \in F$, we have $\ell(f) \geq 3$ because G is a connected graph with at least 3 vertices. Using the previous proposition, we conclude that

$$2m = \sum_{f \in F} \ell(f) \geq 3c,$$

and hence $c \leq \frac{2}{3} \cdot m$. Now using Euler's Theorem, we have

$$\begin{aligned} m + 2 &= n + c \\ &\leq n + \frac{2}{3} \cdot m. \end{aligned}$$

It follows that

$$\frac{1}{3} \cdot m \leq n - 2,$$

and hence $m \leq 3n - 6$. □

Corollary 6.7.7. *K_5 is not planar.*

Proof. Notice that K_5 has 5 vertices and $\binom{5}{2} = 10$ edges. Since $3 \cdot 5 - 6 = 9$, it follows from Proposition 6.7.6 that K_5 is not planar. □

Proposition 6.7.8. *Let G be a connected planar graph with n vertices, m edges, and no triangles (i.e. no 3-cycles). If $n \geq 3$, then $m \leq 2n - 4$.*

Proof. For each face, we must have at least 4 edges on its boundary, so

$$2m = \sum_{f \in F} \ell(f) \geq 4c,$$

and hence $c \leq \frac{1}{2} \cdot m$. Now using Euler's Theorem, we have

$$\begin{aligned} m + 2 &= n + c \\ &\leq n + \frac{1}{2} \cdot m. \end{aligned}$$

It follows that

$$\frac{1}{2} \cdot m \leq n - 2,$$

and hence $m \leq 2n - 4$. □

Corollary 6.7.9. *$K_{3,3}$ is not planar.*

Proof. Notice that $K_{3,3}$ is bipartite so has no triangles. Now $K_{3,3}$ has 6 vertices and $3 \cdot 3 = 9$ edges. Since $2n - 4 = 2 \cdot 6 - 4 = 8$, it follows from Proposition 6.7.8 that $K_{3,3}$ is not planar. □

Proposition 6.7.10. *If G is a finite planar graph, then there exists $v \in V$ with $d(v) \leq 5$.*

Proof. Since a graph is planar if and only if each of its connected components is planar, it suffices to prove the result for *connected* finite planar graphs. So suppose that G is a connected finite planar graph with n vertices and m edges. If $n \leq 2$, then this result is trivial. Suppose then that $n \geq 3$. If $d(v) \geq 6$ for all $v \in V$, then

$$2m = \sum_{v \in V} d(v) \geq 6n,$$

so $m \geq 3n$, which contradicts Proposition 6.7.6. Therefore, there must exist $v \in V$ with $d(v) \leq 5$. □

Proposition 6.7.11. *$\chi(G) \leq 6$ for every finite planar graph G.*

Proof. The proof is by induction on the number of vertices of G. Notice that if G has one vertex, then the result is trivial. Suppose then that $\chi(G) \leq 6$ for all finite planar graphs G with n vertices. Consider an arbitrary finite planar graph with $n+1$ vertices. By Proposition 6.7.10, we can fix a vertex v with $d(v) \leq 5$. Now the graph $G - v$ is planar and has n vertices, so by induction we know that $\chi(G-v) \leq 6$. Fix a proper 6-coloring $c\colon V\backslash\{v\} \to [6]$ of $G - v$. Now v has at most 5 neighbors in G, so we can fix $i \in [6]$ such that $c(w) \neq i$ for all w adjacent to v. Thus, if we extend c by letting $c(v) = i$ for some such i, then we have a proper coloring of G using at most 6 colors. The result follows by induction. □

In fact, with a little more work, we can prove the following.

Theorem 6.7.12. $\chi(G) \leq 5$ *for every planar graph* G.

Proof. The proof is by induction on the number of vertices of G. Notice that if G has one vertex, then the result is trivial. Suppose then that $\chi(G) \leq 6$ for all finite planar graphs G with n vertices. Consider an arbitrary finite planar graph with $n+1$ vertices. By Proposition 6.7.10, we can fix a vertex v with $d(v) \leq 5$. Now the graph $G - v$ is planar and has n vertices, so by induction we know that $\chi(G-v) \leq 6$. Fix a proper 6-coloring $c\colon V\backslash\{v\} \to [5]$ of $G - v$. Now if there is an $i \in [5]$ such that $c(w) \neq i$ for all w adjacent to v, then we obtain a proper 5-coloring of G as in the previous proposition.

Suppose then that all 5 colors occur on the neighbors of v. In some planar embedding of G, call the neighbors w_1, w_2, w_3, w_4, w_5 in a clockwise circle around v. Consider the subgraph $H_{1,3}$ of $G - v$ induced by the vertices currently colored 1 and 3. If w_1 and w_3 are in different connected components of $H_{1,3}$, then we can switch the colors 1 and 3 in the connected component of w_1 (so in particular w_1 is now colored 3), which then allows us to obtain a proper coloring of G with 5 colors by coloring v with 1.

Suppose then that w_1 and w_3 are in the same connected component of $H_{1,3}$. We can then fix a w_1, w_3-path P in $H_{1,3}$ of vertices alternating in color between 1 and 3. We now try the same strategy with w_2 and w_4 by considering the subgraph $H_{2,4}$ of $G - v$ induced by the vertices currently colored 2 and 4. Notice that w_2 and w_4 cannot be in the same connected component of $H_{2,4}$, because otherwise we would have w_2, w_4-path in $H_{2,4}$ of vertices alternating in color between 2 and 4, which would have to cross P, contradicting planarity (notice that the paths can't cross at a vertex because the vertices on the paths have different colors). Since w_2 and w_4 are not in the same connected component of $H_{2,4}$, we can switch the colors 2 and 4 in the connected component of w_2 (so in particular w_2 is now colored 4), which then allows us to obtain a proper coloring of G with 5 colors by coloring v with 2. The result follows by induction. □

In fact, the following much (much) harder result is true.

Theorem 6.7.13 (Four Color Theorem, Appel-Haken)**.** $\chi(G) \leq 4$ *for every planar graph* G.

We now examine convex regular polyhedra, which are 3-dimensional shapes each of whose faces is a convex polygon with the same number of edges, and such that the number of faces that meet at any point is equal throughout. These convex regular polyhedra can be viewed as graphs embedded on a sphere, but notice that a graph can be embedded on the plane exactly when it can be embedded on the sphere. One can see this by picking a point on the sphere (not hit by any vertex/edge) and doing a stereographic projection. Thus, we can study these polyhedra by studying planar graphs such that $d(v)$ is constant for all $v \in V$, and $\ell(f)$ is constant for all $f \in F$. Let d be the common degree of vertices, and let ℓ be the common length of faces. Notice that $d \geq 3$, that $\ell \geq 3$, and that

$$dn = 2m = \ell c.$$

Now using Euler's Theorem, we conclude that

$$\begin{aligned} 2 &= n - m + c \\ &= \frac{2m}{d} - m + \frac{2m}{\ell}, \end{aligned}$$

and hence

$$\frac{1}{d}+\frac{1}{\ell}=\frac{1}{2}+\frac{1}{m}.$$

Since $\frac{1}{m}>0$, it follows that

$$\frac{1}{d}+\frac{1}{\ell}>\frac{1}{2}.$$

Now we know that $d\geq 3$, so if $\ell\geq 6$, then we would have

$$\frac{1}{d}+\frac{1}{\ell}\leq\frac{1}{3}+\frac{1}{6}=\frac{1}{2},$$

which is a contradiction. Similarly, we know that $\ell\geq 3$, so if $d\geq 6$, then we would have

$$\frac{1}{d}+\frac{1}{\ell}\leq\frac{1}{6}+\frac{1}{3}=\frac{1}{2},$$

which is a contradiction. Therefore, we must have $3\leq d\leq 5$ and $3\leq\ell\leq 5$. Furthermore, we can not have *both* $d\geq 4$ and $\ell\geq 4$ because this would imply that

$$\frac{1}{d}+\frac{1}{\ell}\leq\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

which is a contradiction.

This gives the following possible pairs (d,ℓ):

$$(3,3)\quad(3,4)\quad(3,5)\quad(4,3)\quad(5,3).$$

Now from d and ℓ, we can compute m using the equation

$$\frac{1}{d}+\frac{1}{\ell}=\frac{1}{2}+\frac{1}{m}.$$

Furthermore, once we have m as well, then we can determine n and c from the equation

$$dn=2m=\ell c.$$

Doing all the calculations, we conclude the following:

- $d=3$ and $\ell=3$: We then have $m=6$, so $n=4$ and $c=4$.
- $d=3$ and $\ell=4$: We then have $m=12$, so $n=8$ and $c=6$.
- $d=3$ and $\ell=5$: We then have $m=30$, so $n=20$ and $c=12$.
- $d=4$ and $\ell=3$: We then have $m=12$, so $n=6$ and $c=8$.
- $d=5$ and $\ell=3$: We then have $m=30$, so $n=12$ and $c=20$.

d	ℓ	m	n	c	poly
3	3	6	4	4	tetrahedron
3	4	12	8	6	cube
3	5	30	20	12	dodecahedron
4	3	12	6	8	octahedron
5	3	30	12	20	icosahedron

6.8 Ramsey Theory

We begin with the following result.

Proposition 6.8.1. *Given any coloring of the edges of K_6 using two colors, there always exists a triangle such that all edges have the same color.*

Proof. Consider an arbitrary coloring of the edges of K_6 with two colors. Call the colors red and blue. Pick an arbitrary vertex u. Since u is incident to 5 total edges, either u is incident to at least 3 red edges or u is incident to at least 3 blue edges. We now have two cases:

- *Case 1:* Suppose that u is incident to at least 3 red edges. Fix 3 such edges, and call the other endpoints of these edges w_1, w_2, and w_3. Now if the 3 edges with endpoints amongst the w_i are all blue, then w_1, w_2, and w_3 form a blue triangle. On the other hand, if any edge with endpoints w_i and w_j where $i \neq j$ is red, then we obtain a red triangle by looking at the vertices u, w_i, and w_j.
- *Case 2:* Suppose that u is incident to at least 3 blue edges. Fix 3 such edges, and call the other endpoints of these edges w_1, w_2, and w_3. Now if the 3 edges with endpoints amongst the w_i are all red, then w_1, w_2, and w_3 form a red triangle. On the other hand, if any edge with endpoints w_i and w_j where $i \neq j$ is blue, then we obtain a blue triangle by looking at the vertices u, w_i, and w_j.

Thus, we always obtain either a red triangle or a blue triangle (or possibly both). □

Is 6 best possible? In other words, is it true that every coloring of the edges of K_6 with two colors always has a monochromatic (i.e. either all red or all blue) triangle? Or is there a coloring of the edges of K_5 with two colors such that there are no monochromatic triangles? It turns out that the latter is true, so 6 is indeed best possible. To see this, consider K_5 with vertex set $[5] = \{1, 2, 3, 4, 5\}$. Think about these vertices as forming an outer 5-cycle in order. Color the edges of this cycle RED and color all edges BLUE. More formally, let

$$\text{RED} = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\}.$$
$$\text{BLUE} = \{\{1,3\},\{1,4\},\{2,4\},\{2,5\},\{3,5\}\}.$$

Since any 3-element subset of the five vertices will contain at least one "consecutive" pair of numbers (where we consider 1 and 5 to be "consecutive"), and at least one that is not "consecutive", it follows that this coloring has no monochromatic triangle.

Can we generalize these ideas? If we color the edges of K_{10}, can we always find four vertices such that all edges between them have the same color, i.e. can we always find a monochromatic (or homogeneous) subgraph isomorphic to K_4? What if we color the edges of K_{20} instead?

Definition 6.8.2. *Let $k, \ell \in \mathbb{N}^+$. Define $R(k, \ell)$ to be the least $n \in \mathbb{N}^+$ (if it exists) such that whenever all of the edges of K_n are colored with either red or blue, either there exists a subset $A \subseteq V$ with $|A| = k$ such that all edges having both endpoints in A are red, or there exists a subset $B \subseteq V$ with $|B| = \ell$ such that all edges with both endpoints in B are blue.*

In other words, $R(k, \ell)$ is the least $n \in \mathbb{N}^+$ such that whenever all of the edges of K_n are colored with either red or blue, either there is a subgraph of K_n isomorphic to K_k in which all edges are red, or there is a subgraph of K_n isomorphic to K_ℓ in which all edges are blue. For example, we have the following simple facts:

- $R(k, 1) = 1$ and $R(1, \ell) = 1$ for all $k, \ell \in \mathbb{N}^+$. To see why $R(k, 1) = 1$, notice that for the trivial graph with vertex set $[1]$ and no edges, we may take $B = \{1\}$, which satisfies the definition. The other is symmetric.

- $R(k,2) = k$ and $R(2,\ell) = \ell$ for all $k, \ell \in \mathbb{N}^+$. To see why $R(k,2) \leq k$, notice that for the given any coloring of the edges of K_k, either there is at least blue edge whose endpoints we can take for B, or all edges are red and we can let $A = [n]$. Also, we have $R(k,2) > k-1$ because if we color all edges of K_{k-1} red, then no such A or B exist. Thus, $R(k,2) = k$, and similarly $R(2,\ell) = \ell$.
- $R(3,3) = 6$ from above.
- In general, we have $R(\ell,k) = R(k,\ell)$ because we can switch the role of red and blue.

It is not at all obvious that $R(k,\ell)$ exists for each $k, \ell \in \mathbb{N}^+$, but we will come up with a recursive upper bound for these values shortly. To prepare for this result, we first prove the following special case.

Proposition 6.8.3. *$R(4,3)$ exists, and in fact $R(4,3) \leq 10$.*

Proof. Consider an arbitrary coloring of the edges of K_{10} with red and blue. Pick an arbitrary vertex u. Since u is incident to 9 total edges, either u is incident to at least 6 red edges or u is incident to at least 4 blue edges (otherwise, u would be incident to at most $5+3 = 8$ many edges). We now have two cases:

- *Case 1:* Suppose that u is incident to at least 6 red edges. Fix 6 such edges, and call the other endpoints of these edges $w_1, w_2, \ldots, w_6$. Now look at the subgraph of K_{10} induced by these 6 vertices. Since $R(3,3) = 6$ from above, either there is a red triangle amongst these vertices, or there is a blue triangle amongst these vertices. If there is a blue triangle, then we are done by letting B be the corresponding three vertices. Otherwise, there is a red triangle amongst the w_i, and then we can include u with the 3 vertices that make up the red triangle to obtain a subset $A \subseteq [10]$ with $|A| = 4$ such that all edges having both endpoints in A are red.
- *Case 2:* Suppose that u is incident to at least 4 blue edges. Fix 4 such edges, and call the other endpoints of these edges w_1, w_2, w_3, w_4. Now if the 6 edges with endpoints amongst the w_i are all red, then we can take $A = \{w_1, w_2, w_3, w_4\}$. Otherwise, there is a blue edge amongst the w_i, and we can include u with 2 endpoints of this edge to obtain a set $B \subseteq [10]$ with $|B| = 3$ such that all edges having both endpoints in B are blue.

This completes the proof. □

This idea generalizes to the next inductive argument.

Theorem 6.8.4. *For all $k, \ell \in \mathbb{N}^+$, we have that $R(k,\ell)$ exists, and in fact $R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$.*

Proof. We prove this by induction on the value of $k+\ell$. We know from above that $R(k,1)$ and $R(k,2)$ exist for all $k \in \mathbb{N}^+$, and also that $R(1,\ell)$ and $R(2,\ell)$ exist for all $\ell \in \mathbb{N}^+$. Now let $k, \ell \in \mathbb{N}$ with $k, \ell \geq 2$ and assume that both $R(k-1,\ell)$ and $R(k,\ell-1)$ exist. Define the following numbers:

- $c = R(k-1,\ell)$.
- $d = R(k,\ell-1)$.
- $n = R(k-1,\ell) + R(k,\ell-1) = c+d$.

We show that if we color all of the edges of K_n with red/blue, then either there exists a subset $A \subseteq V$ with $|A| = k$ such that all edges having both endpoints in A are red, or there exists a subset $B \subseteq V$ with $|B| = \ell$ such that all edges with both endpoints in B are blue.

Consider then an arbitrary coloring of the edges of K_n with red and blue. Pick an arbitrary vertex u. Since u is incident to $n-1 = c+d-1$ total edges, either u is incident to at least c red edges or u is incident to at least d blue edges (otherwise, u would be incident to at most $c-1+d-1 = c+d-2 = n-2$ many edges). We now have two cases:

- *Case 1:* Suppose that u is incident to at least c red edges. Fix c such edges, and call the other endpoints of these edges $w_1, w_2, \ldots, w_c$. Now look at the subgraph of K_n induced by these c vertices. Since $c = R(k-1, \ell)$, either there is a subset A of these vertices with $|A| = k-1$ such that all edges having both endpoints in A are red, or there exists a subset B of these vertices with $|B| = \ell$ such that all edges with both endpoints in B are blue. In the latter case, we are done by taking B. In the former case, we can let $A' = A \cup \{u\}$ and notice that $|A'| = k$ and A' has the required properties.
- *Case 2:* Suppose that u is incident to at least d blue edges. Fix d such edges, and call the other endpoints of these edges $w_1, w_2, \ldots, w_d$. Now look at the subgraph of K_n induced by these d vertices. Since $d = R(k, \ell-1)$, either there is a subset A of these vertices with $|A| = k$ such that all edges having both endpoints in A are red, or there exists a subset B of these vertices with $|B| = \ell - 1$ such that all edges with both endpoints in B are blue. In the former case, we are done by taking A. In the latter case, we can let $B' = B \cup \{u\}$ and notice that $|B'| = \ell$ and B' has the required properties.

This completes the proof. □

Notice that using this result, we can immediately conclude that

$$R(4,3) \leq R(3,3) + R(4,2) = 6 + 4 = 10$$

as we showed in the special case above. Working in the other direction, we have the following.

Proposition 6.8.5. $R(4,3) \geq 9$.

Proof. We exhibit a coloring of the edges of K_8 with red and blue such that there is no red K_4 and no blue K_3. Take K_8 an label the vertices clockwise with the numbers from [8]. Color the edge $\{i,j\}$ with $i < j$ blue if $j - i \in \{1, 4, 7\}$, and color it red otherwise. Thus, we color $\{i,j\}$ with $i < j$ red if $j - i \in \{2, 3, 5, 6\}$.

We first argue that there is no blue triangle. Suppose one exists, and let the smallest vertex be i. We would then need to choose two of the three potential vertices $\{i+1, i+4, i+7\}$ (where we "wrap around" beyond 8 if necessary) to add to it. This is impossible, because $4 - 1 = 3$, $7 - 4 = 3$, and $7 - 1 = 6$.

We now argue that there is no red K_4. Suppose one exists, and let the smallest vertex be i. We would then need to choose three of the four potential vertices $\{i+2, i+3, i+5, i+6\}$ to add to it. This is impossible, because we can't choose both $i+2$ and $i+3$, and we also can't choose both $i+5$ and $i+6$. □

In fact, we can improve the upper bound $R(4,3) \leq 10$ with a little work.

Proposition 6.8.6. $R(4,3) \leq 9$.

Proof. In the proof of Proposition 6.8.3, we argued that in an arbitrary red/blue coloring of the edges of K_{10}, if we take an arbitrary vertex u, then either u is incident to at least 6 red edges or u is incident to at least 4 blue edges. We now argue that in an arbitrary red/blue coloring of the edges of K_9, there exists a vertex u that is incident to either at least 6 red edges of at least 4 blue edges. From here, we can follow the proof of Proposition 6.8.3.

Suppose then that we have an arbitrary red/blue coloring of the edges of K_9. Since every vertex is incident to 8 edges, if there is no vertex u that is incident to either at least 6 red edges of at least 4 blue edges, then every vertex must be incident to exactly 5 red edges and exactly 3 blue edges. Thus, if we look at the subgraph of K_9 containing all of the vertices but only the red edges, then each of the 9 vertices has degree 5. Thus, the resulting graph would have an odd number of vertices of odd degree, which is a contradiction. □

Combining the two previous results, we conclude that $R(4,3) = 9$. Using Theorem 6.8.4, it follows that

$$\begin{aligned} R(4,4) &\leq R(3,4) + R(4,3) \\ &= 9 + 9 \\ &= 18. \end{aligned}$$

In fact, it can be shown that $R(4,4) = 18$. To conclude this, we need to show that there is a coloring of the edges K_{17} with red/blue such that there is no monochromatic K_4. This is possible as follows. Given $i, j \in \{0, 1, 2, \ldots, 16\}$ with $i < j$, color the edge $\{i, j\}$ with $i < j$ red if $j - i$ is a quadratic residue modulo 17 (i.e. if some perfect square has $j - i$ as a remainder upon division by 17). In other words, we color $\{i, j\}$ red if $j - i \in \{1, 2, 4, 8, 9, 13, 15, 16\}$, and blue otherwise. Using some number theory, one can show that this coloring has the required properties.

Here is a table with many of the known values of $R(k, \ell)$, along with bounds on the numbers that we still do not know.

$k\backslash\ell$	2	3	4	5	6	7
2	2	3	4	5	6	7
3	3	6	9	14	18	23
4	4	9	18	25	35-41	49-61
5	5	14	25	43-49	58-87	80-143
6	6	18	35-41	58-87	102-165	113-298
7	7	23	49-61	80-143	113-298	205-540

In addition to the results of this table, it is also known that $R(3, 8) = 28$ and $R(3, 9) = 36$. However, the best current bounds for $R(3, 10)$ are that $40 \leq R(3, 10) \leq 43$. Asymptotically, it is known that $R(3, k) \approx \frac{k^2}{\log k}$.

Theorem 6.8.7. *For any $k, \ell \in \mathbb{N}$ with $k, \ell \geq 2$, we have*

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}.$$

Proof. We prove the result by induction on the value of $k + \ell$.

- *Base Case:* Suppose that $k + \ell = 4$. We then have that $k = 2$ and $\ell = 2$. Now $R(2, 2) = 2$, and we have
$$\binom{2 + 2 - 2}{2 - 1} = \binom{2}{1} = 2.$$
Thus, the statement is true when $k + \ell = 4$.

- Assume that $m \geq 4$ and that we know that the statement is true whenever $k + \ell = m$. Suppose now that we have values of $k, \ell \in \mathbb{N}$ with $k \geq 2$, $\ell \geq 2$, and $k + \ell = m + 1$. Notice that if $k = 2$, then $R(2, \ell) = \ell$ and
$$\binom{2 + \ell - 2}{2 - 1} = \binom{\ell}{1} = \ell,$$
so the statement is true. Also, if $\ell = 2$, then $R(k, 2) = k$ and
$$\binom{2 + k - 2}{2 - 1} = \binom{k}{1} = k,$$
so the statement is true. Suppose now that $k \geq 3$ and $\ell \geq 3$. We then have that $k - 1 \geq 2$, that $\ell - 1 \geq 2$, that $(k-1) + \ell = m$, and that $k + (\ell - 1) = m$. Therefore, using Theorem 6.8.4 and induction, we have
$$\begin{aligned} R(k, \ell) &\leq R(k - 1, \ell) + R(k, \ell - 1) \\ &\leq \binom{(k-1) + \ell - 2}{(k-1) - 1} + \binom{k + (\ell - 1) - 2}{k - 1} \\ &= \binom{k + \ell - 3}{k - 2} + \binom{k + \ell - 3}{k - 1} \\ &= \binom{k + \ell - 2}{k - 1}, \end{aligned}$$

where the last line follow from Proposition 5.2.1.

The result follows by induction. □

Corollary 6.8.8. *For any $k \in \mathbb{N}$ with $k \geq 2$, we have*

$$R(k,k) \leq \binom{2k-2}{k-1}.$$

Proof. Immediate from the previous result. □

Corollary 6.8.9. *For any $k \in \mathbb{N}$ with $k \geq 2$, we have $R(k,k) \leq 4^{k-1}$.*

Proof. Let $k \in \mathbb{N}$ with $k \geq 2$. We know from Corollary 5.2.3 that

$$\binom{2k-2}{0} + \binom{2k-2}{1} + \cdots + \binom{2k-2}{k-1} + \cdots + \binom{2k-2}{2k-2} = 2^{2k-2}.$$

Since every term in the above sum is nonnegative, it follows that

$$\begin{aligned}\binom{2k-2}{k-1} &\leq 2^{2k-2} \\ &= (2^2)^{k-1} \\ &= 4^{k-1}.\end{aligned}$$

The result now follows from the previous corollary. □

Theorem 6.8.10. *For all $k \geq 3$, we have $R(k,k) > 2^{k/2}$.*

Proof. Let $n \in \mathbb{N}^+$, and suppose that we color the edges of K_n with two colors randomly, by flipping a fair coin for each edge. Given a subset S of $[n]$ with $|S| = k$, what is the probability that S is monochromatic? There are $2^{\binom{k}{2}}$ many ways to color the edges with endpoints in S, and only two of them are monochromatic. Thus, the probability that S is monochromatic is

$$\frac{2}{2^{\binom{k}{2}}}.$$

But this value is just the probability that our one particular set S is monochromatic. What is the probability that *some* such S is monochromatic? There are $\binom{n}{k}$ many subsets S, so since the probability of the union is bounded by the sum of the probabilities, the probably that *some* subset of $[n]$ of size k is monochromatic is at most

$$\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}}.$$

Now if this number is less than 1, then we know that there is some positive probability that no monochromatic set of size k exists. So is the above value less than 1? If $n \leq 2^{k/2}$, then we have

$$\begin{aligned}\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}} &< \frac{n^k}{k!} \cdot \frac{2}{2^{\binom{k}{2}}} \\ &\leq \frac{2n^k}{k! \cdot 2^{\binom{k}{2}}} \\ &\leq 2 \cdot \frac{n^k}{k! \cdot 2^{k(k-1)/2}} \\ &\leq 2 \cdot \frac{2^{k^2/2}}{k! \cdot 2^{k(k-1)/2}}.\end{aligned}$$

Since

$$\frac{k^2}{2} - \frac{k(k-1)}{2} = \frac{k}{2}(k-(k-1)) = \frac{k}{2},$$

we get

$$\begin{aligned}\binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}} &< 2 \cdot \frac{2^{k^2/2}}{k! \cdot 2^{k(k-1)/2}} \\ &= \frac{2 \cdot 2^{k/2}}{k!},\end{aligned}$$

which is certainly less than 1 if $k \geq 3$ (by a trivial induction). □