Homework 7: Due Monday, March 2

Problem 1:
a. For each $n \in \mathbb{N}^+$, give an example of a set $\mathcal{F} \subseteq \mathcal{P}(\{1, 2, \ldots, n\})$ with $|\mathcal{F}| = 2^{n-1}$ and such that $A \cap B \neq \emptyset$ whenever $A, B \in \mathcal{F}$. That is, $\mathcal{F}$ should be a collection of 2^{n-1} many subsets of $\{1, 2, \ldots, n\}$ with the property that any pair of sets from $\mathcal{F}$ have a common element.
b. Let $n \in \mathbb{N}^+$. Let $\mathcal{F} \subseteq \mathcal{P}(\{1, 2, \ldots, n\})$ be such that $|\mathcal{F}| > 2^{n-1}$, i.e. suppose that $\mathcal{F}$ is a collection of more than 2^{n-1} many subsets of $\mathcal{P}(\{1, 2, \ldots, n\})$. Show that there exists two elements of $\mathcal{F}$ that are disjoint.

Problem 2: Show that if A and B are countable sets, then $A \times B$ is countable.

Problem 3: Show that the set $\mathbb{R} \backslash \mathbb{Q}$ or all irrational numbers is uncountable.

Problem 4:
a. Recall that $\{0, 1\}^*$ is the set of all finite sequences of 0's and 1's (of any finite length). Show that $\{0, 1\}^*$ is countable.
b. Let S be the set of all infinite sequences of 0's and 1's (so an element of S looks like $11000101110\ldots$). Show that S is uncountable.

Problem 5:
a. As in Problem 4b, let S be the set of all infinite sequences of 0's and 1's. Show that there exists a bijection $f \colon \mathcal{P}(\mathbb{N}) \to S$.
b. Carefully explain why Problem 4b and part a, taken together, imply that $\mathcal{P}(\mathbb{N})$ is uncountable.

Problem 6: Using the digits 1 through 9 only (so exclude 0), how many 13 digits numbers are there in which no two consecutive digits are the same? Explain your reasoning.