

Homework 6: Due Wednesday, February 26

Problem 1: Using the function $d: \mathbb{N}^+ \rightarrow \mathbb{N}$ from Definition 3.3.4, show that

$$\{n \in \mathbb{N}^+ : d(n) \text{ is odd}\} = \{n^2 : n \in \mathbb{N}^+\}.$$

Problem 2: Let A, B, C be sets and let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. Show that if $g \circ f$ is surjective and g is injective, then f is surjective.

Problem 3: A *lattice point* in the plane is a point of the form (a, b) where $a, b \in \mathbb{Z}$. For example, $(3, 5)$ is a lattice point but $(\pi, 1)$ is not. Show that given any 5 lattice points in the plane, there exists two of the points whose midpoint is also a lattice point.

Problem 4:

- For each $n \in \mathbb{N}^+$, give an example of a set $S \subseteq \{1, 2, \dots, 2n\}$ with $|S| = n$ such that $\gcd(a, b) > 1$ for all $a, b \in S$ with $a \neq b$.
- Let $n \in \mathbb{N}^+$. Suppose that $A \subseteq \{1, 2, \dots, 2n\}$ and $|A| = n + 1$. Show that there exists $a, b \in A$ with $a \neq b$ such that $\gcd(a, b) = 1$.

Problem 5: Let $n \in \mathbb{N}^+$. Given any $n + 2$ many integers, show that it is always possible to find two of them such that either their sum or their difference (or both) is divisible by $2n$.

Problem 6: Suppose that you have a group of 10 people and that the age of every person in the group is between 1 and 100 (inclusive). Suppose also that all of the ages are distinct, so there are not two people of the same age.

- Show that there exist two nonempty distinct subsets A and B of people such that the sum of the ages of the people in A equals the sum of the ages of the people in B .
- Show moreover that you can find A and B as in part a that are also *disjoint*, i.e. for which no person is in both A and B .

Example: Suppose that the ages of the people in the group are 3, 7, 13, 19, 24, 30, 38, 49, 63, 78. One such example is $A = \{3, 13, 78\}$ and $B = \{7, 19, 30, 38\}$ since $3 + 13 + 78 = 94 = 7 + 19 + 30 + 38$.

Hint: How many possible nonempty subsets of people are there? What's the largest possible sum?